BUNDLE COMPLEXES AND BORDISM OF IMMERSIONS

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ABSTRACT

The work of this thesis divides into two parts. The object of the first part is to show how configuration spaces may be regarded as classifying spaces for immersions in good position. This is done by introducing the notion of transverse bundle complex. The purpose of the second part is to compute the bordims groups of immersions of oriented n-manifolds into $\mathbb{R}^{n+k}$ for $n-2\leq k\leq n$. to discuss the behaviour of double points in these dimensions and to study the forgetful homomorphism $I\Omega_{n,k} \to \Omega_n$ that retains the oriented bordism class of a class of immersions for $n-3\leq k\leq n$.

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Introduction

This work divides into two parts. U. Koschorke and B. Sanderson [K-S] showed configuration spaces may be regarded as classifying spaces for immersions. The object of the first part is to prove a refinement of this result. In order to do this we introduce the notion of <u>transverse bundle complex</u> which generalizes that of transverse CW-complex of [B-R-S].

We start in §1 by proving a result about transversality of maps from differentiable manifolds into vector bundles. Although this result is well known no proof of it seems to have appeared in the literature. Next we define complex bundles and transverse bundle complexes, and show that homotopy classes of maps from differentiable manifolds to transverse bundle complexes have transverse representatives.

In §2 we exhibit the configuration spaces models $C_m(M\xi)$ of May and Segal as bundle complexes. Here $M\xi$ denotes the Thom space of an Euclidean vector bundle ξ. A slight deformation is introduced to $C_m(M\xi)$ to obtain a transverse bundle complex model for $\Omega^m S^m(M\xi)$.

In §3 we study multiple points of selftransverse immersions and show that $\Omega^m S^m(M\xi)$ serves as classifying space for "immersions in good position". The cases $\Omega^2 S^3$ and $\Omega^\infty S^\infty(M\xi)$ were treated in [San] and [Si], respectively, using regular homotopy theory.

Let $I\Omega_{n,k}$ denote the bordism group of immersions of oriented n-manifolds into $\mathbb{R}^{n+k}$ and let $f: I\Omega_{n,k} \to \Omega_n$ denote the forgetful homomorphism that retains the oriented bordism class of the domain of a class of immersions. The purpose of the second part of the thesis is to compute the groups $I\Omega_{n,k}$ for $n-2\le k\le n$, to discuss the behaviour of double points in these dimensions and to study the homomorphism $f: I\Omega_{n,k} \to \Omega_n$ for $n-3\le k\le n$. Our main tools are the exact sequences ($n<2k-1$)

$$\cdots \to \Omega^{\zeta_k}_{n-k} \xrightarrow{\partial} I\Omega_{n,k} \xrightarrow{f} \Omega_n \xrightarrow{S} \Omega^{\zeta_k}_{n-k-1} \to \cdots$$

obtained by Szűcs [Sz] and Koschorke [K], and

$$\cdots \to \Omega^{\Delta SO}_{n-k} \xrightarrow{\delta} I\Omega_{n,k} \xrightarrow{g} I\Omega_{n,k+1} \xrightarrow{e} \Omega^{\Delta SO}_{n-k-1} \to \cdots$$

obtained by Salomonsen [Sal]. $\Omega_i^{\zeta}k$ denotes the bordism group of i-manifolds with a certain $\mathbb{Z}_2 \times SO(k)$ structure on the stable normal bundle and $\Omega_i^{\Delta SO}$ denotes the bordism group of i-manifolds whose stable normal bundle splits as a sum of a bundle with itself.

The bordism groups of immersions were studied first by Wells [We] who determined the unoriented groups $IN_{n,n}$ and $IN_{4n,4n-1}$. These results were extended by Koschorke and Olk who completed the computations of $IN_{n,k}$ for $n-2\le k\le n$. See [K, §10]. We shall make use of these computations.

A sketch of how these sequences are obtained is given in §4. In particular, the upper sequence is established using a transverse bundle complex. In §5 we use the techniques of [K,§9] to calculate the group $\Omega_i^{\zeta}k$, $\Omega_i^{\Delta SO}$ for $0\le i\le 2$.

In §7 we study the homomorphism $I\Omega_{n,k} \to \Omega_n$ for $k = n-3, n-2$. The groups $I\Omega_{n,k}$ $(k \geq n-2)$ are determined except for some extensions problems in some few cases. Double points of oriented immersions and its relation to bordism groups of oriented embeddings are investigated in §8.

CONTENTS

PART I

§1 TRANSVERSALITY AND BUNDLE COMPLEXES

We show how some results about transversality in the smooth category can be generalized. All vector bundles are assumed to be Euclidean. If ξ is a vector bundle we let $D\xi, S\xi$ and $M\xi$ denote the associated disc bundle, sphere bundle and Thom space, respectively.

Let $\xi \xrightarrow{p} X$ be a vector bundle and $\delta: X \to \mathbb{R}^+$ a strictly positive map. The $\xi_{-\delta}$ bundle neighbourhood of X will be the open set $\{v \in \xi \mid \delta p(v) > |v|\}$. $\xi_{-\delta}$ has a natural vector bundle structure such that $\xi_{-\delta} \cong \xi$.

Let M be a smooth manifold and ξ a vector bundle over some space X. A map $f: M \to \xi$ is said to be transverse to X if $f^{-1}X$ is a smooth submanifold of M with a smooth tubular neighbourhood ν such that $f|\nu$ is a vector bundle map onto some bundle neighbourhood $\xi_{-\delta}$ of X.

A fibre bundle ξ over a space X is of finite type if X may be covered by a finite number of open sets $U_1, \ldots, U_k$ with $\xi|U_i$ trivial.

It is well known that any fibre bundle over a finite dimensional CW complex is of finite type+.

Proposition 1 Let ξ be a vector bundle of finite type over a space X, M a paracompact manifold and $f: M \to \xi$ any map. Then f is homotopic to a map g transverse to X. Further, if $f|U$ is already transverse to X for U an open neighbourhood of A closed in M, then the homotopy can be taken fixed on a neighbourhood of A.

Proof. Special case $\xi = \mathbb{R}^k$.

There is a vector bundle map

$$\begin{tikzcd} \nu \arrow[r, "f"] \arrow[d, "p"'] & \mathbb{R}^k_{-\delta} \arrow[d] \\ f^{-1}0 \cap U \arrow[r] & 0 \end{tikzcd}$$

where $f^{-1}0 \cap U$ is a smooth submanifold of U with a tubular neighbourhood ν and $\mathbb{R}^k_{-\delta}$ is the open ball of radious δ centred at 0. We can define a Euclidean structure on ν by making the composite $\nu \to \mathbb{R}^k_{-\delta} \cong \mathbb{R}^k$ an Euclidean vector bundle map.

+ See J. Milnor, Differential topology, mimeographed, Princeton University, 1958.

We may consider $f|\nu$ as a section S_0 of the dual vector bundle $\mathrm{Hom}(\nu, \mathbb{R}^k_{-\delta}) \to f^{-1}0 \cap \nu$. Let S_1 be a smooth section close enough to S_0 such that for each $t \in I$ the section $(1-t)S_0 + t S_1$ induces a vector bundle map. This can be achieved since the subspace $\mathrm{Iso}(\nu, \mathbb{R}^k_{-\delta})$ is open.

Let V, W, Y be open sets satisfying

$$A \subset Y \subset \overline{Y} \subset W \subset \overline{W} \subset V \subset \overline{V} \subset U$$

Let $\lambda : \mathbb{R}^+ \to I$ and $l : f^{-1}0 \cap U \to I$ be maps with

$$\lambda(t) = \begin{cases} 1 & t \leq 1 \\ 0 & t \geq 2 \end{cases}$$

$$l(m) = \begin{cases} 1 & m \notin W \\ 0 & m \in \overline{Y} \end{cases}$$

If $f_1 : \nu \to \mathbb{R}^k_{-\delta}$ denotes the vector bundle map associated to the section S_1, then we define $h: M \to \mathbb{R}^k$ by

$$h(x) = \begin{cases} f(x) & , x \notin \nu \\ \dfrac{|x|(1-\lambda|x|Ip(x))f(x)+\lambda|x|Ip(x)f_1(x)}{|(1-\lambda|x|Ip(x))f(x)+\lambda|x|Ip(x)f_1(x)|} & , x \in \nu \end{cases}$$

then

i) $h \simeq f$ rel $\overline{Y}$

ii) $h(x) = f_1(x)$ if $x \in (M-W) \cap D\nu$

iii) $h|U$ is transverse to 0.

We are now in position to apply the smooth transversality theorem to $h|M-\overline{W}$, since $h|(M-\overline{W}) \cap (D\nu - S\nu)$ is differentiable and transverse to 0. It follows then that there exists a map $g\colon M - \overline{W} \to \mathbb{R}^k$ such that $g \simeq h$ $\text{rel}(M-\overline{W}) \cap \overline{V}$ with $g^{-1}0$ a smooth submanifold and g transverse to 0. g can be extended to M by setting $g(x) = h(x)$, $x \in W$, proving the proposition for the special case.

General case: We will prove the result by induction on the number of elements of an open cover $\{B_1,..,B_n\}$ of X with $\xi|B_i \cong B_i \times \mathbb{R}^k$.

If ξ is trivial then the composite

$$M \xrightarrow{f} \xi \overset{\psi}{\cong} X \times \mathbb{R}^k \xrightarrow{\pi} \mathbb{R}^k$$

is homotopic to a map h with h transverse to 0 and the homotopy can be taken fixed in a neighbourhood of A. There is then a bundle map

$$\begin{tikzcd} \nu \arrow[r, "h|"] \arrow[d, "p"'] & \mathbb{R}^k_{-\delta} \arrow[d] \\ h^{-1}0 \arrow[r] & 0 \end{tikzcd}$$

where ν is a tubular neighbourhood of $h^{-1}0$ which is given a Euclidean structure as before. Finally define g by

$$g(x) = \begin{cases} (1-\lambda|m|)f(m) + \lambda|m|\,\psi^{-1}(p\,f(m), h(m)) & m \in \nu \\ f(m) & m \notin \nu \end{cases}$$

Assume now the proposition holds for bundles with n coordinate charts and let $\xi \xrightarrow{p} X$ be a vector bundle with $X = B_1 \cup \ldots \cup B_{n+1}$ and $\xi|B_i \cong B_i \times \mathbb{R}^k$, B_i open in X. Let V an open set with $\overline{V} \subset f^{-1}(\xi|B_1 \cup \ldots \cup B_n)$ and $V \cup f^{-1}(\xi|B_{n+1}) = M$ Make $f|f^{-1}(\xi|B_1 \cup \ldots \cup B_n)$ homotopic to a map h such that h is transverse to $B_1 \cup \ldots \cup B_n$ and the homotopy is fixed on a neighbourhood W of $A \cap f^{-1}(\xi|B_1 \cup \ldots \cup B_n)$.

Let $\{\lambda_0, \lambda_1 : M \to I\}$ be a partition of unity subordinate to $\{f^{-1}(\xi|B_1 \cup \ldots \cup B_n), M - \overline{V}\}$. Define $h' : M \to \xi$ by

$$h'(m) = \lambda_0(m) h(m) + \lambda_1(m) f(m),$$

then

$$h'(m) = \begin{cases} h(m) & , m \in \overline{V} \\ f(m) & , \left\{ \begin{array}{l} m \in M - f^{-1}(\xi|B_1 \cup \ldots \cup B_n) \\ \text{or } \; m \in W \end{array} \right. \end{cases}$$

But $h'|f^{-1}(\xi|B_{n+1})$ is homotopic (rel a neighbourhood of $(A \cup \overline{V}) \cap f^{-1}(\xi|B_{n+1})$) to a map $g : f^{-1}\xi|B_{n+1} \to \xi|B_{n+1}$ with g transverse to B_{n+1}. $h \cup g : V \cup f^{-1}(\xi|B_n) = M \to \xi$ is then the required map.□

Corollary 2. Let M, ξ be as in proposition 1 and $f: M \to \xi$ a map such that $f|\partial M \to \xi$ is transverse to X. Then there exists a map g homotopic to f such that $g|\partial M = f|\partial M$ and g is transverse to X.

Proof. This follows since ∂M is always closed in M and using a collar we may assume f is transverse to X in a neighbourhood of X. □

As a straight forward consequence we have the well known generalization of Thom's theorem. (See [St, p.18])

Corollary 3. Let M be a closed manifold and ξ a vector bundle over some space X. There is a one-to-one correspondence between the homotopy classes of maps $M \to M\xi$, and bordism classes of closed submanifolds of M with a ξ-structure on their normal bundles.

Proof. This follows in the usual way by observing that any map $f: M \to \xi$ factors through $\xi| K \subset \xi$ for some compact $K \subset X$. □

A <u>bundle complex structure</u> on a space X is a filtration $X_0 \subset X_1 \subset \ldots \subset X$ such that

i) For $n \geq 1$ X_n is obtained from X_{n-1} by attaching n-dimensional disc bundles along the sphere bundles.

ii) X has the weak topology with respect to the family $\{X_0, X_1, \ldots\}$.

Since disjoint unions of n-dimensional vector bundles are themselves vector bundles we will assume that X_n is obtained from X_{n-1} by attaching at most one n-dimensional disc bundle $D\xi^n \to B_n$ along its sphere bundle $S\xi$. The map

$$h_n : D\xi \to X_n \subset X$$

will be called the <u>characteristic map for</u> X_n. Note that $X_n - X_{n-1} \cong D\xi - S\xi$ and $X_n/X_{n-1} \cong M\xi$.

We will simply say that a space X is a <u>bundle complex</u> when some particular bundle complex structure on X is understood. In general, X admits many structures.

For instance, any space X is a trivial bundle complex with

$$X_0 = X_1 = \ldots = X.$$

Let M be a smooth manifold and X a bundle complex. A map $f\colon M \to X$ is said to be <u>transverse</u> if for each n either $f^{-1}(X_n - X_{n-1}) = \phi$ or there is a commutative diagram

$$\begin{tikzcd}
 & X & \\
\bar{T}_n \arrow[ur, "f"] \arrow[rr, "t_n"'] \arrow[d] & & D\xi^n \arrow[ul, "h_n"'] \arrow[d] \\
f^{-1}B_n \arrow[rr] & & B_n
\end{tikzcd}$$

where h_n is the characteristic map for X_n, $T_n = f^{-1}(X_n - X_{n-1})$, $\bar{T}_n$ is a codimension 0 submanifold (with boundary) of M, $f^{-1}B_n$ is a submanifold of T_n having $\bar{T}_n$ as a disc neighbourhood, and the square is a disc bundle map.

We say that a bundle complex X is a <u>transverse bundle complex</u> if for each n, k, $0 < k < n$, either $h_n^{-1}(X_k - X_{k-1}) = \phi$ or there are fibre subbundles

$$Y_{n,k} \subset T_{n,k} \subset S\xi$$

with fibres $N \subset \nu^k$, where N is a k-codimensional submanifold of S^{n-1} with a tubular neighbourhood ν and there is a commutative diagram

$$\begin{tikzcd}
 & X_{n-1} & \\
\bar{T}_{n,k} \arrow[ur, "h_n"] \arrow[rr, "t_{n,k}"] \arrow[d] & & D\xi^k \arrow[ul, "h_k"'] \arrow[d] \\
Y_{n,k} \arrow[rr] & & B_k
\end{tikzcd}$$

where h_n, h_k are characteristic maps, $T_{n,k} = h_n^{-1}(X_k - X_{k-1})$, $\bar{T}_{n,k} \to Y_{n,k}$ is induced by $\bar{\nu}^k \to N$ and the bottom square is a disc bundle map.

Examples

i) A Thom space is a bundle complex with

$$* = X_0 = X_1 = \ldots = X_{n-1} , M\xi = X_n = X_{n+1} = \ldots .$$

ii) The geometric realization of a simplicial space has a natural bundle complex structure.

iii) Any transverse CW-complex in the sense of [B-R-S] is a transverse bundle complex with X_n = the n-skeleton obtained from X_{n-1} by attaching a n-dimensional disc bundle over a discrete space.

Theorem 4 Suppose X is a transverse bundle complex, M^n a compact n-manifold and $f: M \to X$ a map with $f|\partial M$ transverse. Then there exists a transverse map $g: M \to X$ with $g \simeq f$ rel$(\partial M \cup f^{-1}X_0)$.

Proof. Since M is compact and X has the weak topology with respect to $\{X_0, X_1, \ldots\}$ we can assume $\operatorname{im} f \subset X_j$. We will prove the result by induction on such j. For $j = 0$ the result is trivial.

Now assume $\operatorname{im} f \subset X_j, j \geq 1$. Let $c: \partial M \times I \to M$ be a collar for ∂M and by a previous homotopy assume that f is constant on collar lines. Since $X_j - X_{j-1} \cong D\xi^j - S\xi^j$ the base space B_j of ξ^j is embedded in X via the zero section. Apply proposition 1 to make f transverse to B_j by a homotopy rel im $c \cup f^{-1}X_{j-1}$. Hence we have a commutative diagram

$$\begin{tikzcd}
 & X & \\
\overline{f^{-1}h_j(\bar{\xi}_\varepsilon^j)} \arrow[ur, "f"] \arrow[rr, "t_i"] \arrow[d] & & \bar{\xi}_\varepsilon^j \arrow[ul, "h_j|"'] \arrow[d] \\
f^{-1}B_j \arrow[rr, "f"'] & & B_j
\end{tikzcd}$$

where h_j is the characteristic map for X_j, $\bar{\xi}_\varepsilon^j$ is a bundle neighbourhood of B_j, $\overline{f^{-1}h_j(\xi_\varepsilon^j)}$ is a codimension zero submanifold of M and the bottom square is a disc bundle map.

If $j > n$, then im f does not meet $h_j(\bar{\xi}_\varepsilon^j)$ and since

$f|\mathrm{im}\, c$ is already transverse, $f(\mathrm{im}\, c) \subset X_{j-1}$. Hence we can project radially $\mathrm{im}\, f$ into X_{j-1} to get a map $h\colon M \to X$ with $h \simeq f$ rel $f^{-1}X_j$ and $\mathrm{im}\, h \subset X_{j-1}$. The result now follows by induction assumption.

If $j \leq n$, since $f|\mathrm{im}\, c$ is already transverse we compose $f|M-\mathrm{im}\, c$ with a standard homotopy of X_j in itself (obtained by expanding $\bar{\xi}^j_\varepsilon$ onto $\bar{\xi}^j$ and using the attaching map $h_j\colon \bar{\xi}^j \to X_{j-1}$) and then extend using the collar to a homotopy of f which keeps $\partial M \cup f^{-1}X_{j-1}$ fixed. Therefore we have a commutative diagram

$$\begin{tikzcd}
 & X & \\
\overline{f^{-1}(X-X_{j-1})} \arrow[ur, "f"] \arrow[rr, "t_j"'] \arrow[d] & & \bar{\xi}^j \arrow[ul, "h_j"'] \arrow[d] \\
f^{-1}B_j \arrow[rr] & & B_j
\end{tikzcd}$$

with $f^{-1}(X-X_{j-1})$ a codimension zero submanifold of M and the bottom square a disc bundle map. Let $M_0 = f^{-1}(X_{j-1})$ and $\delta = t_j^{-1}(S\xi^j)$. M_0 is a manifold (with corners) with boundary $\delta \cup (\partial M \cup M_0)$. $f|\partial M \cap M_0$ is already transverse. We show now that we can assume up to a homotopy $\mathrm{rel}(f^{-1}X_{j-1})$ that $f|\delta$ is as well transverse. The result follows now by induction.

We need to show that $f|\delta$ can be assumed to be transverse. There is a commutative diagram

$$\begin{tikzcd}
& X_{j-1} & \\
\delta \arrow[ur, "f"] \arrow[rr, "t_j"] \arrow[d] & & S\xi^j \arrow[ul, "h_j"'] \arrow[d] \\
f^{-1}B_j \arrow[rr] & & B_j
\end{tikzcd}$$

where the bottom square is a sphere bundle map. By means of a collar and proposition 1 we may homotope f and assume

t_j has a smooth mapping transformations[+]. Transversality of X (as a bundle complex) now implies transversality of

$f|:\delta \to X_{j-1}$.□

Remark. The notion of transversality for bundle complexes is very restrictive. An alternative and much simpler approach to transverse bundle complexes can be given if we restrict to bundle complexes built with smooth vector bundles. A bundle complex is then said to be transverse if the attaching maps

$$h_j: S\xi^j \to X_{j-1}$$

are transverse. Theorem 4 follows essentially in the same manner.

+ See N. Steenrod, The topology of fibre bundles , Princeton University Press, 1951, for the definition of mapping transformations of a fibre bundle map.

§2 A TRANSVERSE BUNDLE COMPLEX MODEL FOR $\Omega^m S^m(M\xi)$

We begin this section by showing that the well known configuration space model $C_m(M\xi)$ of May and Segal for $\Omega^m S^m(M\xi)$ is a bundle complex. As it fails to be transverse, a slight deformation is introduced to $C_m(M\xi)$ in order to obtain a transverse bundle complex model for $\Omega^m S^m(M\xi)$.

Let $\tilde{C}_{m,k} = \{(z_1,..,z_k) : z_i \in \mathbb{R}^m$ and $z_i \neq z_j$ for $i \neq j\}$. $\tilde{C}_{m,k}$ is an open subspace of $\mathbb{R}^{mk}$ on which the symmetric group Σ_k acts freely. For a based space $(X,*)$ let Σ_k act on X^k by permuting coordinates.

Recall that $C_m(X) = (\coprod_{k\geq 1} \tilde{C}_{m,k} \times_{\Sigma_k} X^k)/\sim$, where $\sim$ is the equivalence relation generated by

$$[z_1,\ldots,z_k;x_1,\ldots,x_k] \sim [z_1,\ldots,z_{k-1};x_1,\ldots,x_{k-1}] \text{ if } x_k = *$$

and $[z_1,x_1] \sim *$ if $x_1 = *$.

Let $\Omega^\infty S^\infty X$ denote the direct limit $\lim\limits_{m\to\infty} \Omega^m S^m X$. It is well known that the configuration space $C_m(X)$ ($m = \infty$ allowed) is w.h.e. to $\Omega^m S^m X$ provided X is a connected compactly generated Hausdorff space with non-degenerate base point. There is a natural filtration

$$*: F_0C_m(X) \subset F_1C_m(X) \subset \ldots \subset F_\infty C_m(X) = C_m(X)$$

where $F_jC_m(X) = (\coprod_{k=1}^{j} \tilde{C}_{m,k} \times_{\Sigma_k} X^k)/\sim$. We will simply write F_j instead of $F_jC_m(X)$ when the reference to m and X is understood.

Lemma 5. <u>For</u> $1 \leq j \leq \infty$ <u>there are homeomorphisms</u>

$$I_j : F_jC_m(X/A) \to (\coprod_{k=1}^{j} \tilde{C}_{m,k} \times_{\Sigma_k} X^k)/\approx$$

<u>where</u> $\approx$ <u>is the equivalence relation generated by</u>

$$[z_1,\ldots,z_k;x_1,\ldots,x_k] \approx [z_1,\ldots,z_{k-1};x_1,\ldots,x_{k-1}] \text{ if } x_k \in A$$

and $[z_1;x_1] \approx *$ if $x_1 \in A$.

Proof.- Consider the identification maps

$$\begin{tikzcd}
\coprod_{k=1}^{j} \tilde{C}_{m,k} \times_{\Sigma_k} X^k \arrow[r, "q"] \arrow[d, "p"'] & \coprod_{k=1}^{j} \tilde{C}_{m,k} \times_{\Sigma_k} (X/A)^k \arrow[r, "q'"] & F_jC_m(X/A) \arrow[dll, dashed, "I_j"] \\
(\coprod_{k=1}^{j} \tilde{C}_{m,k} \times_{\Sigma_k} X^k)/\sim & &
\end{tikzcd}$$

where q is induced by $X \to X/A$. Clearly p and $q'\circ q$ have the same fibres and therefore there exists a homeomorphism l_j making the new diagram commutative$\square$

Lemma 6 $C_m(X)$ <u>has the weak topology with respect to the family</u> $(F_jC_m(X))_{j\geq 1}$.

Proof.- Assume $U \subset C_m(X)$ with $U \subset F_j$ open in F_j for $j \geq 1$. There are commutative diagrams ($j \geq 1$)

$$\begin{array}{ccc}
\coprod_{k=1}^{\infty} \tilde{C}_{m,k} \times_{\Sigma_k} X^k & \supset & \coprod_{k=1}^{j} \tilde{C}_{m,k} \times_{\Sigma_k} X^k \\
\Big\downarrow p_\infty & & \Big\downarrow p_j \\
C_m(X) & \supset & F_jC_m(X)
\end{array}$$

where p_∞ and p_j are the quotient maps. Then for $1 \leq k \leq j$

$$p_j^{-1}(U \cap F_j) \cap \tilde{C}_{m,k} \times_{\Sigma_k} X^k = (p_\infty^{-1}U) \cap \tilde{C}_{m,k} \times_{\Sigma_k} X^k$$

is open in $\tilde{C}_{m,k} \times_{\Sigma_k} X^k$ and thus $p_\infty^{-1} U$ is open in $\coprod_{k=1}^{j} \tilde{C}_{m,k} \times_{\Sigma_k} X^k$. The result now follows since p_∞ is a quotient map$\square$

If ξ is any vector bundle over a space B, we let $\xi_{m,k}$ denote the associated twisted power bundle

$$\tilde{C}_{m,k} \times_{\Sigma_k} (\xi)^k \to \tilde{C}_{m,k} \times_{\Sigma_k} B^k.$$

If ξ is n-dimensional, then $\xi_{m,k}$ is nk-dimensional, and there is a homeomorphism

$$\psi_k : \tilde{C}_{m,k} \times_{\Sigma_k} (D\xi)^k \to D\xi_{m,k}.$$

It follows from lemma 5 that

$$F_j C_m(M\xi) \cong \left(\coprod_{k=1}^{j} D\xi_{m,k}\right)/\sim$$

where $\sim$ is generated by

$\psi_k[z_1,\ldots,z_k;x_1,\ldots,x_k] \sim \psi_{k-1}[z_1,\ldots,z_{k-1},x_1,\ldots,x_{k-1}]$ if $x_k \in S\xi$ and $[z_1;x_1] \sim *$ if $x_1 \in S\xi$.

Let $g_j : S\xi_{m,j} \to F_{j-1} C_m(M\xi)$ be defined by

$$g_j\psi_j[z_1,\ldots,z_j,x_1,\ldots,x_j] = \psi_{j-1}[z'_1,\ldots,z_{j-1},x_1,\ldots,x_{j-1}]$$

if $x_j \in S\xi$

then

$$F_j C_m(M\xi) \cong F_{j-1} C_m(M\xi) \cup_{g_j} D\xi_{m,j}$$

Hence we have

Proposition 7. <u>Let</u> ξ <u>be an n-dimensional vector bundle and let</u> $X_j = F_k C_m(M\xi)$ <u>if</u> $nk \leq j < n(k+1)$. <u>Then the filtration</u>

$$* = X_0 \subset X_1 \subset \ldots \subset C_m(M\xi)$$

<u>induces a natural bundle complex structure on</u> $C_m(M\xi)$.□

Let $d^s\colon D\xi \to D\xi$ $(0 \leq s \leq 1)$ be defined by

$$d^s(v) = \begin{cases} (1+s)v & ,|v| \leq 1/1+s \\ v/|v| & ,|v| \geq \frac{1}{1+s} \end{cases}$$

We will simply write d instead of d^1. $\{d^s\}_{0\leq s\leq 1}$ define a homotopy starting at $1_{D\xi}$ and ending at d.

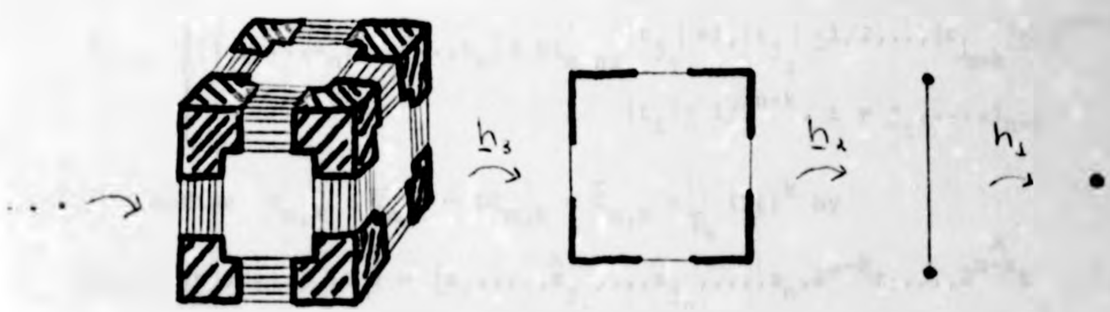
Now define $\underline{C_m}(M\xi) = (\coprod_{k\geq 1} \bar{C}_{m,k} \times_{\Sigma_k} (D\xi)^k)/\approx$ where $\approx$ is the equivalence relation generated by

$[z_1,\ldots,z_k; t_1,\ldots,t_k] \approx [z_1,\ldots,z_{k-1}; dt_1,\ldots,dt_{k-1}]$ if $t_k \in S\xi$, and $[z_1;t_1] \approx *$ if $t_1 \in S\xi$.

Notice that if $\alpha = [z_1,\ldots,z_n;t_1,\ldots t_n] \in \bar{C}_{m,n} \times_{\Sigma_n} (D\xi)^n$ then

$$\alpha \approx \begin{cases} [z_1,\ldots,z_{n-1}; 2t_1,\ldots,2t_{n-1}] & \text{if } |t_n| = 1, |t_i| \leq 1/2,\ i \neq n. \\ [z_1,\ldots,z_{n-2}; 4t_1,\ldots,4t_{n-2}] & \text{if } |t_n| = 1, |t_{n-1}| \geq 1/2, |t_i| \leq 1/4\ \ i \neq n, n-1 \\ \cdot & \\ \cdot & \\ \cdot & \\ [z_1, 2^{n-1}t_1] & \text{if } |t_n| = 1, |t_{n-1}| \geq 1/2, |t_{n-2}| \geq 1/4, \ldots, |t_2| \geq 1/2^{n-2}, |t_1| \leq 1/2^{n-1} \\ * & \text{if } |t_n| = 1, |t_{n-1}| \geq 1/2, |t_{n-2}| \geq 1/4, \ldots, |t_1| \geq 1/2^{n-1} \end{cases}$$

Example $\underline{C_m}(S^1)$



$\underline{C_m}(\xi)$ is also naturally filtered by

$$\{\underline{F_k C_m}(M\xi) = (\coprod_{j=1}^{k} C_{m,j} \times_{\Sigma_j} (D\xi)^j)/\approx\}$$

We will write $\underline{F_k}$ instead of $\underline{F_k C_m}(M\xi)$ whenever m and ξ are understood.

Proposition 8 $\underline{C_m}(M\xi)$ <u>is a transverse bundle complex</u>.

Proof. $\underline{F_n} \cong \underline{F_{n-1}} \cup_{g_n} D\xi_{m,n}$ where $g_n : S\xi_{m,n} \to \underline{F_{n-1}}$ is given by

$[z_1,\ldots,z_n;t_1,\ldots,t_n] \mapsto [z_1,\ldots,z_{n-1},dt_1,\ldots,dt_{n-1}]$ if $|t_n|=1$.

That $\underline{C_m}(M\xi)$ has the weak topology with respect to $\{\underline{F_k}\}$ follows as in proposition 7. This shows that $\underline{C_m}(M\xi)$ is a bundle complex.

For $0<k<n$ let $T_{n,k} = g_n^{-1}(F_k - F_{k-1})$. Then

$$\bar{T}_{n,k} = \left\{ [z_1,\ldots,z_n;t_1,\ldots,t_n] \in D\xi_{m,n} : \begin{array}{l} |t_{j_1}| = 1, |t_{j_2}| \geq 1/2, \ldots, |t_{j_{n-k}}| \geq 1/2^{n-k-1} \\ |t_i| \leq 1/2^{n-k},\ i \neq j_1,\ldots,j_{n-k} \end{array} \right\}$$

Define $\theta_{n,k} : \bar{T}_{n,k} \to D\xi_{m,k} = \tilde{C}_{m,k} \times_{\Sigma_k} (D\xi)^k$ by

$[z_1,\ldots,z_n;t_1,\ldots,t_n] \mapsto [z_1,\ldots,\hat{z}_{j_1},\ldots,\hat{z}_{j_n},\ldots,z_n,2^{n-k}t_1,\ldots,2^{n-k}\hat{t}_{j_1}\ldots]$,

where as usual "^" means delete.

We have then a commutative diagram

$$\begin{tikzcd}
 & F_{n-1} & \\
\overline{T}_{n,k} \arrow[ur, "h_n"] \arrow[r, "\Theta_{n,k}"'] \arrow[d] & D\xi_{m,k} = \tilde{C}_{m,k} \times_{\Sigma_k} (D\xi)^k \arrow[u, "h_k"'] \arrow[d, "1 \times_{\Sigma_k} \pi^k"] & \\
\overline{Y}_{n,k} \arrow[r] & B_{m,k} = \tilde{C}_{m,k} \times_{\Sigma_k} B^k &
\end{tikzcd}$$

where $\overline{Y}_{n,k} = \left\{ \begin{array}{l} [z_1,\ldots,z_n; t_1,\ldots,t_n] \in \overline{T}_{n,k} \,\middle|\, |t_{j_1}| = 1,\ldots,|t_{j_{n-k}}| \geq 1/2^{n-k-1} \\ \qquad\qquad\qquad |t_i| = 0 \text{ for } i \neq j_1,\ldots,j_{n-k} \end{array} \right\}$

and h_n, h_k are characteristic maps.

Clearly $\Theta_{n,k}$ is a disc bundle map and therefore $\underline{C_m}(M\xi)$ is a transverse bundle complex $\square$

Let Σ^n act on $(D\xi)^n$ by permuting coordinates. We now define inductively a family of equivariant maps $I_n: (D\xi)^n \to (D\xi)^n$ and equivariant homotopies $H_n: 1_{(D\xi)^n} \simeq I_n$.

Lemma 9. <u>For any bundle ξ the inclusion of</u>

$\partial((D\xi)^n/\Sigma_n)=\{[v_1,\dots,v_n]\in(D\xi)^n/\Sigma_n : |v_i|=1$ for some $i\}$ <u>into</u>

$(D\xi)^n/\Sigma_n$ <u>is a cofibration</u>.

Proof. Define a norm $\| \quad \| : (D\xi)^n/\Sigma_n \to I$ by

$\| [v_1,\dots,v_n] \| = \mathrm{Max}\{|v_i|\}$.

Let $h = 1 - \| \quad \| : (D\xi)^n/\Sigma_n \to I$ and let $U = h^{-1}[0,1)$.

Then $\partial((D\xi)^n/\Sigma_n) = h^{-1}0$ and therefore has U as a collaring.

Let $\phi : U \times I \to (D\xi)^n/\Sigma_n$ be defined by

$\phi(\alpha,S) = [v_1/_{1-S+S\|\alpha\|},\dots,v_n/_{1-S+S\|\alpha\|}]$ if $\alpha = [v_1,\dots,v_n]$.

Then $\phi(\alpha,0)=\alpha$, $\|\phi(\alpha,1)\| = \mathrm{Max}\{\frac{|v_i|}{\|\alpha\|}\} = \frac{\|\alpha\|}{\|\alpha\|} = 1$ and if

$\|\alpha\|=1$ then $\phi(\alpha,S) = \alpha$. Therefore $\partial((D\xi)^n/\Sigma_n)$ is a zero

set and a strong collaring deformation retract in $(D\xi)^n/C_n$.

Hence $\partial((D\xi)^n/\Sigma_n) \subset (D\xi)^n/\Sigma_n$ is a cofibration.□

Let $I_1 : D\xi \times I \to D\xi$ be the projection. Assume we have

defined a Σ_{n-1}-equivariant map $I_{n-1} : (D\xi)^{n-1} \to (D\xi)^{n-1}$ and

a Σ_{n-1}-equivariant homotopy $H_{n-1} : (D\xi)^{n-1} \times I \to (D\xi)^{n-1}$ starting at $1_{(D\xi)^{n-1}}$

and ending at I_{n-1}. Define $G_n : \partial(D\xi)^n \times I \to \partial(D\xi)^n$ by

$$G_n(t_1,\dots,t_n;S) = \begin{cases} (d^{2S}(t_1),\dots,d^{2S}(t_n)) & 0 \le S \le 1/2 \\ (x_1,\dots,x_{j-1},t_j,x_j,\dots,x_{n-1}) & 1/2 \le S \le 1 \end{cases}$$

where $(x_1,\dots,x_{n-1}) = H_{n-1}(dt_1,\dots,\widehat{dt_j},\dots,dt_n;2S-1)$ and $|t_j|=1$.

We have then a commutative diagram

$$\begin{tikzcd}
\partial(D\xi)^n \times 0 \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \partial(D\xi)^n \times I \arrow[d, phantom, "\cap"] \arrow[ddr, bend left, "G_n"] & \\
(D\xi)^n \times 0 \arrow[r, phantom, "\subset"] \arrow[drr, bend right, "1_{(D\xi)^n}"'] & (D\xi)^n \times I \arrow[dr, dashed, "H_n"'] & \\
 & & (D\xi)^n
\end{tikzcd}$$

with G_n Σ_n-equivariant. By the previous lemma there exists a Σ_n-equivariant homotopy H_n making commutative the new diagram.□

Theorem 10. <u>There is a weak homotopy equivalence</u>

$$\phi : \underline{C}_m(M\xi) \to C_m(M\xi)$$

Proof. We define a sequence of maps $\phi_n : \underline{F}_n \to F_n$ and $\psi_n : F_n \to \underline{F}_n$ by the commutativity of the diagram

$$\begin{tikzcd}
F_n \arrow[dr, "\psi_n"'] & D\xi_{m,n} \arrow[l, "q"'] \arrow[r, "f_n"] \arrow[d, "p"] & F_n \\
& \underline{F}_n \arrow[ur, "\phi_n"'] &
\end{tikzcd}$$

where p and q are the canonical quotients and $f_n[z_1,\dots,z_n; t_1,\dots,t_n] = [z_1,\dots,z_n; I_n(t\,,\dots,t_n)]$. ϕ_n is well defined, for if $p[z_1,\dots,z_n; t_1,\dots,t_n] = p[z'_1,\dots,z'_n; t'_1,\dots,t'_n]$ then $z_1 = z'_{i_1},\dots, z_{n-1} = z'_{i_{n-1}}$, $dt_1 = dt'_{i_1},\dots,dt_{n-1} = dt'_{i_{n-1}}$, and $|t_n| = |t'_{i_n}| = 1$.

So $f_n[z_1,\dots,z_n; t_1,\dots,t_n] = [z_1,\dots,z_n; I_n(t_1,\dots,t_n)]$

$$= [z'_{i_1},\dots,z'_{i_n}; I_{n-1}(dt_1,\dots,dt_{n-1}), t'_{i_n})$$

$$= f_n[z'_1,\dots,z'_n; t'_1,\dots,t'_n].$$

Similarly ψ_n is also well defined. Moreover, there are commutative diagrams

$$\begin{array}{ccc} \underline{F}_n & \xrightarrow{\phi_n} & F_n \\ \cap & & \cap \\ \underline{F}_{n+1} & \xrightarrow[\phi_{n+1}]{} & F_{n+1} \end{array} \qquad\qquad \begin{array}{ccc} F_n & \xrightarrow{\psi_n \circ d} & \underline{F}_n \\ \cap & & \cap \\ F_{n+1} & \xrightarrow[\psi_{n+1}]{} & \underline{F}_{n+1} \end{array}$$

Since $\underline{C}_m(M\xi)$ has the weak topology with respect to the family $\{\underline{F}_k\}$ it follows that there exists a map $\phi : \underline{C}_m(M\xi) \to C_m(M\xi)$ such that $\phi|\underline{F}_k = \phi_k$. ϕ_n and ψ_n are homotopy inverses, the homotopies being induced by H_n. The result now follows since any map $S^k \to \underline{C}_m(M\xi)$ factors through some $\underline{F}_n$. $\square$

§3 $\Omega^m S^m(M\xi)$ AS A CLASSIFYING SPACE FOR IMMERSIONS.

U. Koschorke and B. Sanderson [K-S,thm.1.1] showed how configuration spaces may be regarded as classifying spaces for immersions. The object of this section is to prove a refinement of this result using our transverse bundle complex model $\underline{C}_m(M\xi)$. Other results in this direction can be found in [Sand] , [Si] using different techniques.

We now recall some basic properties about wreath products. If G is any group, then the wreath product $\Sigma_n \wr G$ is $\Sigma_n \times G^n$ as a set. If $(\sigma;g_1,\ldots,g_n)$ and $(\tau;h_1,\ldots,h_n) \in \Sigma_n \wr G$, then their product is defined to be $(\sigma\tau;g_{\tau(1)}h_1,\ldots,g_{\tau(n)}h_n)$. $\Sigma_n \wr G$ acts freely on the acyclic space $E\Sigma_n \times (EG)^n$ by

$$(x;\ y_1,\ldots,y_n)\cdot(\sigma;g_1,\ldots,g_n) = (x\sigma;\ y_{\sigma(1)}g_1,\ldots,y_{\sigma(n)}g_n).$$

The quotient space $E\Sigma_n \times (EG)^n/\Sigma_n \wr G$ is homeomorphic to $E\Sigma_n \times_{\Sigma_n} (BG)^n$, where Σ_n acts on $(BG)^n$ by permuting coordinates. It follows that

$$B(\Sigma_n \wr G) \simeq E\Sigma_n \times_{\Sigma_n} (BG)^n.$$

If G is a subgroup of the orthogonal group $O(k)$, then $\Sigma_n \wr G$ can be represented as the matrix subroup of $O(kn)$ which consists of all Σ_n-permutation matrices obtained from

$$\begin{pmatrix} A_1 & 0 & \cdot & \cdot & 0 \\ 0 & A_2 & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & A_n \end{pmatrix}$$

where $A_i \in G$.

Throughout this section ξ will denote a fixed j-dimensional vector bundle over a space B.

Proposition 11. <u>If ξ admits a reduction to a subrgroup G of $O(j)$ then $\xi_{m,n}$ admits a reduction to $\Sigma_n \wr G$.</u>

Proof. Let $g : B \to BG$ classify ξ. Then $\xi_{m,n}$ is classified by

$$\bar{C}_{m,n} \times_{\Sigma_n} B^n \xrightarrow{f \times_{\Sigma_n} g^n} E\Sigma_n \times_{\Sigma_n} (BG)^n$$

where f is $\bar{C}_{m,n} \subset \tilde{C}_{\infty,n} \simeq E\Sigma_{n}$. $\square$

Let M^n, N^{n-j} be closed manifolds and $g : N \looparrowright M$ a selftransverse immersion. Let

$\tilde{N}_k = \{(x_1, \ldots, x_k) : x_i \in N,\ \ x_i \neq x_j \text{ if } i \neq j \text{ and } g(x_i) = g(x_j)\}$

$\tilde{N}_k$ is a submanifold of N^k of dimensions $n - kj$. Σ_{k-1} and Σ_k act freely of $\tilde{N}_k$ by permuting the first $(k-1)$ coordinates and all the coordinates, respectively. Let $N'_k = \tilde{N}_k/\Sigma_{k-1}$ and $N_k = \tilde{N}_k/\Sigma_k$. There are canonical immersions

$g'_k : N'_k \looparrowright N$ given by $g'_k\,[x_1, \ldots, x_k] = x_k$ and

$g_k : N_k \looparrowright M$ by $g_k\,[x_1, \ldots, x_k] = g(x_1)$.

Let ν be a normal bundle for g. We have then an immersion $\bar{g} : \nu \looparrowright M$ and a commutative diagram

$$\begin{tikzcd}
\nu \arrow[dr, looparrow, "\bar{g}"] \arrow[dd, "p"'] & \\
& M \\
N \arrow[ur, looparrow, "g"'] &
\end{tikzcd}$$

We now define ν_k' and ν_k to be the quotients of $\overbrace{\nu\times\ldots\times\nu}^{(k-1)\text{-times}} \times N\big|_{\tilde{N}_k}$ under the action of Σ_{k-1} and of $\overbrace{\nu\times\ldots\times\nu}^{k\text{-times}}\big|\tilde{N}_k$ under the action of Σ_k. We have then a bundle monomorphism

$$\begin{array}{ccc} \nu_k' & \xrightarrow{\overline{p_k}} & \nu_k \\ \downarrow & & \downarrow \\ N_k' & \xrightarrow[p_k]{} & N_k \end{array}$$

where p_k is the k-regular covering induced by the identity on representatives. Notice that for all $x \in N_k$ the family $\{\overline{p}_k(\nu_k'|_y)\}_{y\in p_k^{-1}(x)}$ meet transversaly in $\nu_k|x$.

Proposition 12. ν_k' <u>and</u> ν_k <u>are normal bundles for</u> g_k' <u>and</u> g_k.

Proof. Since ν is a normal bundle for g we have an exact sequence of vector bundles over N:

$$0 \to \tau_N \xrightarrow{\alpha} g^*\tau_M \xrightarrow{\beta} \nu \to 0$$

where α is determined by the monomorhphism $g_* : \tau_N \to \tau_M$.

Clearly $\tau_{N_k} = \{[w_1,\dots,w_k] \in (\tau_N \times \dots \times \tau_N|_{\tilde{N}_k})/\Sigma_k : g_* w_1 = \dots = g_* w_k\}$

and $g_k^*(\tau_M) = \{([x_1,\dots,x_k],v) \in N_k \times \tau_M : v \text{ is tangent at } g(x_1)\}$

The exactness of above sequence implies now the exactness of the sequence of vector bundles over N_k

$$0 \to \tau_{N_k} \xrightarrow{\alpha_k} g_k^*\tau_M \xrightarrow{\beta_k} \nu_k \to 0$$

where α_k and β_k are defined as follows:

$\alpha_k[w_1,\dots,w_k] = ([x_1,\dots,x_k], g_* w_1)$ if w_i is a vector at x_i,

and $\beta_k([x_1,\dots,x_k],v) = [\beta(x_1,v),\dots, (x_k,v)]$.

The case for ν_k' is similar.□

Let the Σ_k-bundle $\tilde{N}_k \to N_k$ be classified by

$$\begin{array}{ccc} \tilde{N}_k & \xrightarrow{\bar{c}} & E\Sigma_k \\ \downarrow & & \downarrow \\ N_k & \xrightarrow[c]{} & B\Sigma_k \end{array}$$

If ν admits a ξ-structure, that is, a vector bundle map

$$\begin{tikzcd} \nu \arrow[r, "\tilde{f}"] \arrow[d] & \xi \arrow[d] \\ N \arrow[r, "f"'] & B \end{tikzcd}$$

Then an $E\Sigma_k \times_{\Sigma_k} (\xi)^k$ - structure on ν_k

$$\begin{tikzcd} \nu_k \arrow[r, "\tilde{h}"] \arrow[d] & E\Sigma_k \times_{\Sigma_k} (\xi)^k \arrow[d] \\ N_k \arrow[r, "h"'] & B\Sigma_k \times_{\Sigma_k} B^k \end{tikzcd}$$

is induced, where $h[x_1,\dots,x_k] = [\bar{c}(x_1,\dots,x_k);\ fx_1,\dots,fx_k]$ and $\tilde{h}[v_1,\dots,v_k] = [\bar{c}(pv_1,\dots,pv_k);\ \tilde{f}v_1,\dots,\tilde{f}v_k]$. In particular, if ν admits a reduction to a subgroup G of $O(j)$ then ν_k admits a reduction to $\Sigma_k \wr G$.

If x is a real number let $[x]$ denote the greatest integer less equal to x.

We will say that an immersion $g: N \looparrowright M$ is a ξ-<u>immersion in good position</u> if there is a commutative diagram

$$\begin{tikzcd}
& D\nu \arrow[r] \arrow[dl, "\bar{g}"'] \arrow[dd, "p"] & D\xi \arrow[dd] \\
M & & \\
& N \arrow[r] \arrow[ul, "g"] & B
\end{tikzcd}$$

where ν is a normal bundle for g, $\bar{g}$ is an immersion, the square is a disc bundle map, and for $1 \leq k \leq [n/j]$:

i) There is a commutative diagram of immersions

$$\begin{tikzcd}
N \arrow[r, "g"] & M \\
N'_k \arrow[u, "g'_k"] \arrow[r, "p_k"] \arrow[d] & N_k \arrow[u, "g_k"'] \arrow[d] \\
D\nu'_k \arrow[r, "\bar{p}_k"'] \arrow[uu, bend left, "\bar{g}'_k"] & D\nu_k \arrow[uu, bend right, "\bar{g}_k"']
\end{tikzcd}$$

ii) $\bar{g}(v) = \bar{g}_k[v_1,\ldots,v_{k-1},v]$ if $p(v) = \bar{g}'_k[v_1,\ldots,v_{k-1},n]$, some $n \in N$.

iii) $\{x \in M : |\bar{g}(x)^{-1}| \geq k\} = \operatorname{im} \bar{g}_k$.

Intuitively speaking a ξ-immersion in good position is a selftransverse immersion in which the normal bundle fits nicely at multiple points.

We will consider triples $(N,g,\bar{g})$ where $g: N \looparrowright M$ is a ξ-immersion in good position, $\bar{g} : \nu \to M \times \mathbb{R}^m$ is an embedding satisfying $\Pi_1 \bar{g} = \bar{g}$ and $\Pi_2 \bar{g}(v) = \Pi_2 \bar{g}(pv)$, where $\Pi_1 : M \times \mathbb{R}^m \to M$ and $\Pi_2 : M \times \mathbb{R}^m \to \mathbb{R}^m$ are the projections. We are now in position to state the main result of this section. See [K-S, thm 1.1]

Theorem 13 <u>(Koschorke-Sanderson) There is a bijection between the bordism classes of triples</u> $(N,g,\bar{g})$ <u>and the homotopy classes of maps</u> $M \to \Omega^m S^m(M\xi)$.

Although $C_m(M\xi)$ is not a transverse bundle complex, by means of $C_m(M\xi)$ we can show that any homotopy class of maps $M \to C_m(M\xi)$ has nearly transverse representatives.

Lemma 14 <u>Let</u> W <u>be a compact n-manifold. Any map</u> $f_0 : W \to C_m(M\xi)$ <u>is homotopic to a map</u> f_1 <u>such that for</u>

any $1 \leq k \leq [n/j]$ *there are commutative diagrams*

$$\begin{tikzcd}
 & C_m(M\xi) & \\
\underline{M}_k \arrow[ur, "f_1"] \arrow[rr] \arrow[d] & & \tilde{C}_{m,k} \times_{\Sigma_k} (D\xi)^k \arrow[ul, "h_k"'] \arrow[d] \\
\underline{N}_k \arrow[rr] & & \tilde{C}_{m,k} \times_{\Sigma_k} B^k
\end{tikzcd}$$

where h_k *is the characteristic map for* $\tilde{C}_{m,k} \times_{\Sigma_k} (D\xi)^k$,

$\underline{M}_k = f_1^{-1}(F_k - F_{k-1})$ *is a codimension* 0 *submanifold of* W,

$\underline{N}_k = f_1^{-1}(\tilde{C}_{m,k} \times_{\Sigma_k} B^k/\sim - F_{k-1})$ *is a submanifold of* W *having*

$\underline{M}_k$ *as a cube neighbourhood and the bottom square is a cube bundle map*. *Moreover if* $f_0|\partial W$ *already satisfies this condition, then the homotopy can be taken fixed on a neighbourhood of* ∂M.

Proof of the lemma. Since W is compact there exists K such that im $f_0 \subset F_K$. By theorem 4.1 there is a map $g : M \to \underline{F}_K$ transverse and homotopic to $\psi_K f_0$. There are then commutative diagrams

$$\begin{tikzcd}
 & \underline{F}_K & \\
\overline{T}_k \arrow[ur, "g"] \arrow[r, "t_k"] \arrow[d] & \tilde{C}_{m,k} \times \Sigma_k (D\xi)^k \arrow[u, "\underline{h}_k"'] \arrow[d] \\
Y_k \arrow[r] & \tilde{C}_{m,k} \times \Sigma_k B^k
\end{tikzcd}$$

as described on page 9. Y_{k-1} meets $\partial\overline{T}_k$ transversally, since the fibre over $y \in Y_{k-1} \cap \overline{T}_k$ as an element of the bundle $\overline{T}_{k-1}$ is just a neighbourhood of y in $\partial\overline{T}_k$. Similarly,

$$\left\{ m \in \overline{T}_k : t_k(m) = [x_1,\dots,x_k;v_1,\dots,v_k] \ , \ \begin{matrix} |v_i| = 0 \ \& \\ |v_j| > 1/2 \end{matrix} \text{-for some } i,j \right\}$$

meets transversally $\partial\overline{T}_k$. We will assume that the union of this submanifold and Y_{k-1} is a smooth submanifold of W. This can be achieved by changing $t_{[n/j]-1}$ through bundle maps, then $t_{[n/j]-2}$ and so on (See proof of proposition 1 §1).

Let $\underline{M}_k = \bigcup_{i=0}^{[n/j]-k} \left\{ m \in T_{k+i} \;\middle|\; \begin{array}{l} t_{k+i}(m)=[x_1,\dots,x_{k+i};v_1,\dots,v_{k+i}] \text{ with} \\ |v_{k+1}|,\dots,|v_{k+i}|>1/2^{k+i-1}; |v_k|,\dots,|v_1|\le 1/2^{k+i-1} \end{array} \right\}$

and let $\underline{M}_0 = M - (\underline{M}_1 \cup \dots \cup \underline{M}_{[n/j]})$. Define $\Theta_k : \underline{M}_k \to \bar{C}_{m,k} \times_{\Sigma_k} (D\xi)^k$ by $\Theta_k(m) = [x_1,\dots,x_k; 2^{k+i-1}v_1,\dots,2^{k+i-1}v_k]$ if $t_{k+i}(m) = [x_1,\dots,x_{k+i};v_1,\dots,v_{k+i}]$ with $|v_{k+1}|,\dots,|v_{k+i}| > 1/2^{k+i-1}; |v_k|,\dots,|v_1| < 1/2^{k+i-1}$

Θ_k is well defined, for if $m \in \underline{M}_k \cap \bar{T}_{k+i} \cap \bar{T}_{k+\ell}$, $i < \ell$, and $t_{k+\ell}(m) = [x_1,\dots,x_{k+\ell};v_1,\dots,v_{k+\ell}]$ with $|v_{k+\ell}|,\dots,|v_{k+1}| \ge 1/2^{k+\ell-1}$ and $|v_k|,\dots,|v_1| \le 1/2^{k+\ell-1}$ then there are $\ell-i$ vectors of $\{v_{k+1},\dots,v_{k+\ell}\}$, $\{v_{k+1},\dots,v_{k+\ell-i}\}$ say with $|v_{k+\ell-i}|=1, |v_{k+\ell-i-1}| \ge 1/2,\dots$ $|v_{k+1}| \ge 1/2^{\ell-i-1}$ and $t_{k+i}(m) = [x_1,\dots,x_{k+i}; 2^{\ell-i}v_1,\dots,2^{\ell-i}v_k]$.

Hence $[x_1,\dots,x_k; 2^{k+\ell-1}v_1,\dots,2^{k+\ell-1}v_k] =$

$$[x_1,\dots,x_k; 2^{k+i-1}.2^{\ell-i}v_1,\dots 2^{k+i-1}.2^{\ell-i}v_k].$$

Moreover, if $m \in \bar{\underline{M}}_k \cap \bar{\underline{M}}_{k-1}$ and $t_{k+i}(m) = [x_1,\dots x_{k+i};v_1,\dots v_{k+i}]$ with $|v_{k+i}|,\dots,|v_{k+1}| \ge 1/2^{k+i-1}; |v_k|,\dots,|v_1| \le 1/2^{k+i-1}$. Since $m \in \bar{\underline{M}}_{k-1}$ there is a vector in $\{v_1,\dots,v_k\}$, say v_k, such that

$|v_k| = \frac{1}{2^{k+i-1}}$. But then $\theta_k(m) = [x_1,\ldots,x_k; 2^{k+i-1}v_1,\ldots,2^{k+i-1}v_k]$

$$\sim [x_1,\ldots,x_{k-1}; 2^{k+i-1}v_1,\ldots,2^{k+i-1}v_{k-1}]$$

$$= \theta_{k-1}(m)$$

It follows then that the composites

$$\underline{M}_k \xrightarrow{\theta_k} C_{m,k} \times_{\Sigma_k} (D\xi)^k \xrightarrow{h_k} C_m(M\xi)$$

glue together to produce a map $W \xrightarrow{f_1} C_m(M\xi)$. Finally $f_1 = \phi_k \circ g$, the homotopy $W \times I \to C_m(M\xi)$ given by

$(m,s) \mapsto [x_1,\ldots,x_k; (1+s)^{k+i-1} v_1,\ldots,(1+s)^{k+i-1}v_k]$ if $m \in T_{k+i}$,

$t_{k+i}(m) = [x_1,\ldots,x_{k+i}; v_1,\ldots,v_{k+i}]$ and $|v_{k+i}|,\ldots,|v_{k+i}| > 1/2^{k+i-1}$,

$|v_k|,\ldots,|v_1| \leq 1/2^{k+i-1}$. □

Let $f: M \to C_m(M\xi)$ be a map satisfying the transversality conditions of the previous lemma. Without loss of generality we can assume that the composites

(15) $$Y_k \to \tilde{C}_{m,k} \times_{\Sigma_k} B^k \to C_{m,k}$$

are smooth maps.

(If $m = \infty$ each of these composites factor through some $C_{M,k} \subset C_{\infty,k}$, as Y_k has compact clousure)

Let $D\nu = \bigcup_{k=1}^{[n/j]} \{(m,x) \in M \times \mathbb{R}^m \mid \theta_k(m) = [x_1,\ldots,x_k;v_1,\ldots,v_k], x_k = x\}$

and $N = \bigcup_{k=1}^{[m/j]} \{(m,x) \in M \times \mathbb{R}^m \mid \theta_k(m) = [x_1,\ldots,x_k;v_1,\ldots,v_k], x_k = x, |v_k| = 0\}$

For $1 \leq k \leq [n/j]$, the projection $\Pi_1| : \{(m,x) \in D\nu \mid m \in \underline{M}_k\} \to \underline{M}_k$ is a k-covering, the fibre over $y \in \underline{M}_k$ being $\{(y,x_1),\ldots,(y,x_k)\}$ where $\theta_k(y) = [x_1,\ldots,x_k;v_1,\ldots v_k]$.

Using (15) it is posible to define smooth local sections, proving $D\nu$, N are smooth submanifolds of $M \times \mathbb{R}^m$.

There is a disc bundle map

$$\begin{tikzcd} D\nu \arrow[r] \arrow[d, "p"'] & D\xi \arrow[d, "p"] \\ N \arrow[r] & B \end{tikzcd}$$

given by $(y,x) \mapsto v$ if $\theta_k(y) = [x_1,\ldots,x_{k-1},x;v_1,\ldots,v_{k-1},v]$,

and $p(y,x) = (m,x)$ where $\theta_k(m) = [x_1,\ldots,x_{k-1},x;v_1,\ldots,v_{k-1},p(v)]$

(Recall that $\underline{M}_k$ has a bundle structure).

Now define $\tilde{g}$ = inclusion $D\nu \to M \times \mathbb{R}^m$,

$$\bar{g} = \pi_1 \tilde{g} : D\nu \to M$$

$$g = \bar{g}|\text{zero section} : N \to M$$

$\bar{g}$ and g are selftransverse immersions, $D\nu$ a disc normal bundle for g. Let $(m,x) \in D\nu$ as above. Then $\pi_2 \tilde{g}(m,x) = x = \pi_2 p(m,x)$.

We show now that g is a ξ-immersions in good position:

Clearly,

$N_k = \{[(m,x_1),\dots,(m,x_k)] : \theta_{k+i}(m) = [x_1,\dots,x_k,x_{k+1},\dots,x_{k+i};b_1,\dots,b_k,v_{k+1},\dots v_{k+i}], b_i \in B\}$

$D\nu_k = \{[(m,x_1),\dots,(m,x_k)] : \theta_{k+i}(m) = [x_1,\dots,x_k,x_{k+1},\dots,x_{k+i};v_1,\dots,v_{k+i}]\}$ and

$D\nu'_k = \{[(m,x_1),\dots,(m,x_k)] : \theta_{k+i}(m) = [x_1,\dots,x_k;x_{k+1},\dots,x_{k+i};v_1,v_2,\dots,v_{k+i}], |v_k| = 0\}$

$\bar{g}_k : D\nu_k \to M$ is defined by $\bar{g}_k[(m,x_1),\dots,(m,x_k)] = m$ and

$\bar{g}'_k : D\nu'_k \to N$ by $\bar{g}'_k[(m,x_1),\dots,(m,x_k)] = (m,x_k)$ if

$\theta_{k+i}(m) = [x_1,\dots,x_{k+i};v_1,v_2,\dots,v_{k+i}]$, $|v_k| = 0$.

The bundle structures on $\underline{M}_k,\dots,\underline{M}_{\lfloor n/j \rfloor}$ make $\bar{g}_k$ and $\bar{g}'_k$ immersions. One easily checks that these immersions satisfy the conditions i,ii,iii) of page 34.

$(N,g,\bar{g})$ defines a map $M \to C_m(M\xi)$ as follows. Let $\bar{g}_0 = 1_M$. M is filtered by

$$\operatorname{im}\bar{g}_{[n/j]} \subset \operatorname{im}\bar{g}_{[n/j]-1} \subset \ldots \subset \operatorname{im}\bar{g}_1 = \operatorname{im}\bar{g} \subset \operatorname{im}\bar{g}_0 = M.$$

Let $\underline{M}_k = \operatorname{im}\bar{g}_k - \operatorname{im}\bar{g}_{k+1}$ and $\underline{N}_k = \operatorname{im} g - \operatorname{im}\bar{g}_{k+1}$. $\underline{M}_k$ is then the total space of a cube bundle over $\underline{N}_k$ wich admits a $\tilde{C}_{m,k} \times_{\Sigma_k} (D\xi)^k$-structure

$$\underline{M}_k \to \tilde{C}_{m,k} \times_{\Sigma_k} (D\xi)^k$$

defined by $\bar{g}_k[v_1,\ldots,v_k] \mapsto [\pi_2\tilde{g}(v_1),\ldots,\pi_2\tilde{g}(v_k);h(v_1),\ldots,h(v_k)]$ where $h\colon D\nu \to D\xi$ is the bundle map inducing the ξ-structure on ν. If $x \in \bar{\underline{M}}_k \cap \bar{\underline{M}}_{k+1}$ then $x = \bar{g}_k[v_1,\ldots,v_k] = \bar{g}_{k+1}[v_1,\ldots,v_k,w]$ say, with $|w|=1$. If follows then that the bundle maps $\underline{M}_k \to C_{m,k} \times_{\Sigma_k} (D\xi)^k \to C_m(M\xi)$ glue together to produce a map $M \to C_m(M\xi)$.

Now proceed as in the usual Thom construction to see that these constructions define the bijection stated in theorem 13.

Some interesting consequences of this theorem can be found in [K-S], [Sand] , [Si].

PART II

§4 TWO EXACT SEQUENCES INVOLVING BORDISM GROUPS OF IMMERSIONS

Using the Pontrjagin-Thom construction and classical results of Hirsch, R. Wells proved [We] that for $k \geq 1$ the bordism group of immersions of oriented n-manifolds into $\mathbb{R}^{n+k}$ is isomorphic to the stable homotopy group $\pi^S_{n+k}(M\gamma^{\sim k})$, where $M\gamma^{\sim k}$ is the Thom-space of the k-dimensional oriented universal vector bundle. We will denote this group by $I\Omega_{n,k}$. Let $f : I\Omega_{n,k} \to \Omega_n$ denote the forgetful homomorphism that retains the oriented bordism class of the domain of a class of immersions and let $g: I\Omega_{n,k} \to I\Omega_{n,k+1}$ be given by composing a representative immersion with the inclusion $\mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$.

The object of this section is to give a sketch of the way Szűcs [Sz] and Salomonsen [Sal] established the exact sequences $(n < 2k - 1)$.

$$(16) \cdots \to \Omega^{\zeta k}_{n-k} \xrightarrow{\partial} I\Omega_{n,k} \xrightarrow{f} \Omega_n \xrightarrow{S} \Omega^{\zeta k}_{n-k-1} \to \cdots$$

(17) $\cdots \to \Omega^{\Delta SO}_{n-k} \xrightarrow{\delta} I\Omega_{n,k} \xrightarrow{g} I\Omega_{n,k+1} \xrightarrow{e} \Omega^{\Delta SO}_{n-k-1} \to \cdots$

$\Omega_i^{\tilde{}k}$ denotes the bordism group of i-manifolds with a $\left\{\begin{pmatrix} A0 \\ 0A \end{pmatrix}, \begin{pmatrix} 0A \\ A0 \end{pmatrix}\right\}_{A \in SO(k)}$ - structure on the stable normal bundle and $\Omega_i^{\Delta SO}$ denotes the bordism group of i - manifolds whose normal bundle splits as the sum of a bundle with itself.

Sequence (16) will be obtained by means of a transverse bundle complex and theorem 13. It is essentially Szűcs construction. Koschorke [K,7.23] obtained a slightly more general sequence using a different technique.

Recall that $\Omega^{\infty}S^{\infty}X$ stands for the direct limit $\lim_{n\to\infty} \Omega^n S^n X$. $\pi_i^S(X)$ is then naturally isomorphic to $\pi_i(\Omega^{\infty}S^{\infty}X)$. A refined version of Wells result follows from theorem 13 using $C_{\infty}(M\tilde{\gamma}^k)$ as a model for $\Omega^{\infty}S^{\infty}(M\tilde{\gamma}^k)$. In fact, it is easy to see that $\pi_{n+k}(F_m C_{\infty}(M\tilde{\gamma}^k))$ is isomorphic to the bordism group of selftransverse immersions of n-manifolds into $\mathbb{R}^{n+k}$ with at most m-fold intersection points (and the same restrictions on the bording immersions).

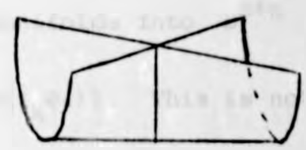
From now on we will assume $n < 2k - 1$. This is the metastable range and corresponds to selftransverse immersions in which only double points are possible. Thus for $n < 2k - 1$

$$I\Omega_{n,k} \cong \Pi_{n+k}(F_2 C_\infty(M\gamma^{-k})) \cong \Pi_{n+k}(\underline{F_2} C_\infty(M\gamma^{-k})).$$

Consider the subgroup of $O(2k)$ consisting of matrices $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ where $A \in SO(k)$. Let ζ_k be the associated universal vector bundle. Note that this subgroup is isomorphic to $\mathbb{Z}_2 \times SO(k)$ and that $\dim \zeta_k = 2k$.

Let Q be the image of the map $\mathbb{R}^{k+1} \to \mathbb{R}^{2k+1}$ given by $(t,u_1,\ldots,u_k) \to (t^2,(t+1)u_1,\ldots,(t+1)u_k,(1-t)u_1,\ldots,(1-t)u_k)$. The above group acts naturally on $\mathbb{R}^{2k+1}$ leaving fixed the first coordinate and Q is invariant under this action.

(t,u)



$(t^2,(t+1)u,(1-t)u)$

$B(\mathbb{Z}_2 \times SO(k))$ can be approximated by compact manifolds $B_1 \subset B_2 \subset \ldots \subset B(\mathbb{Z}_2 \times SO(k))$. Fix an arbitrary N and consider the associated sub-bundle of $\zeta_k \oplus \varepsilon|_{B_N}$ with fibre Q. The intersection of this associated sub-bundle and the sphere bundle $S(\zeta_k \oplus \varepsilon|_{B_N})$ is the image of an immersion with only double points into the manifold $S(\zeta_k \oplus \varepsilon|_{B_N})$. By theorem 13 we obtain a map

$$\rho_N : S(\zeta_k \oplus \varepsilon|_{B_N}) \to F_2 C_\infty(M\tilde{\gamma}^k)$$

The maps ρ_N can be chosen so that $\rho_{N+1}|S(\zeta_k \oplus \varepsilon|_{B_N}) = \rho_N$. Let $\rho : S(\zeta_k \oplus \varepsilon) \to F_2 C_\infty(M\tilde{\gamma}^k)$ be the map defined by $\rho|S(\zeta_k \oplus \varepsilon|_{B_N}) = \rho_n$.

Szűcs showed that for $n < 2k-1$ the bordism group of generic map of oriented n-manifolds into S^{n+k} is isomorphic to $\Pi_{n+k}(F_2 C_\infty(M\tilde{\gamma}^k) \cup_\rho D(\zeta_k \oplus \varepsilon))$. This is not difficult to see if we apply (4)ii to the corresponding transverse bundle complex $\underline{F_2 C_\infty}(M\tilde{\gamma}^k) \cup_\rho D(\zeta_k \oplus \varepsilon)$ and make use of the following well known characterization of generic maps in the metastable range. (See A. Haefliger, Plongements differentiables de varietes dans varietes, Comm. Math. Helv.36(1961),47-82)

Let M^n be a compact manifold and $f : M^n \to U^{n+k}$ a generic map, $n \leq 2k-1$. Then for any point $x \in M$ either

a) f is regular at x and $f^{-1}f(x) = \{x\}$,

b) There exists a unique $y \in M$, $y \neq x$, such that $f(x)=f(y)$, f is regular at x and y and $df(\tau M_x)\oplus df(\tau M_y) = \tau U_{f(x)}$

or

c) Rank $df_x = n-1$, $f^{-1}f(x) = \{x\}$ and it is possible to choose local coordinates on M and U such that f is given by $(t,u_1,\ldots,u_k,v_1,\ldots,v_{n-k-1}) \mapsto (t^2,tu_1,\ldots,tu_k,u_1,\ldots,u_k,v_1,\ldots,v_{n-k-1})$

(See figure on page 45)

Applying the Pontrjagin-Thom construction to the homotopy sequence of the pair

$$(F_2C_\infty(M\tilde{\gamma}^k) \cup_\rho D(\zeta_k\oplus\varepsilon), F_2C_\infty(M\tilde{\gamma}^k))$$

gives sequence (16). We describe now the corresponding homomorphisms.

Given any class $[M] \in \Omega_n$ choose any generic map

$f : M \to S^{n+k}$. Condition c) above implies that the normal bundle of the singular-points-manifold of f admits a $(\zeta_k \oplus \varepsilon)$-structure. $S[M] \in \Omega^{\zeta k}_{n-k-1}$ is represented by this singular-points-manifold of f.

Let $j_k : S^k \to \mathbb{R}^{2k}$ be defined by $j_k(t,u_1,\ldots,u_k) = ((t+1)u_1,\ldots,(t+1)u_k,(1-t)u_1,\ldots,(1-t)u_k)$, where S^k is the unit sphere in $\mathbb{R}^{k+1}$ with coordinates $(t,u_1,\ldots,u_k)$. Note that j_k is an immersion with precisely one double point and the image of j_k is $\left\{\begin{pmatrix} A0 \\ 0A \end{pmatrix}, \begin{pmatrix} 0A \\ A0 \end{pmatrix}\right\}$ -invariant. $\partial : \Omega^{\zeta k}_{n-k} \to I\Omega_{n,k}$ is defined as follows. If Σ represents an arbitrary class in $\Omega^{\zeta k}_{n-k}$ then associated to a tubular neighbourhood of an embedding $\Sigma^{n-k} \subset \mathbb{R}^{n+k}$ there is a fibre bundle with fibre im j_k. The total space of this bundle represents $\partial[\Sigma]$.

Sequence (17) can be obtained as follows. See [Sal] for details. For any $(k+1)$-dimensional vector bundle ξ over a space X, let N^k be the subbundle of $p^*\xi$ consisting of tangents along fibres. There is then a pull-back diagram

(18)
$$\begin{tikzcd} N^k \oplus \varepsilon^1 \arrow[r, dash] \arrow[d, dash] & \xi \arrow[d, "p"] \\ S(\xi) \arrow[r, "p"'] & X \end{tikzcd}$$

and a cofibration

$$S(MN) \cong M(N \oplus \varepsilon^1) \to M\xi \to M\xi \oplus \xi$$

To see this consider the following diagram of pull-backs

$$\begin{tikzcd} R \arrow[r, hook, dash] \arrow[d, hook] & Q \arrow[r, dash] \arrow[d, hook] & S\xi \arrow[d, hook] \\ Q \arrow[r, hook] \arrow[d] & P \arrow[r] \arrow[d] & D\xi \arrow[d] \\ S\xi \arrow[r, hook] & D\xi \arrow[r] & X \end{tikzcd}$$

It is clear then that

$M(N \oplus \varepsilon) \cong Q/R$, $M\xi \cong P/Q$, $P \cong D(\xi \oplus \xi)$ and $(P/Q)/(Q/R) \cong M(\xi \oplus \xi)$.

In particular for $\xi = \tilde{\gamma}^{k+1}$ there is a homotopy equivalence $S(\tilde{\gamma}^{k+1}) \simeq BSO(k)$, therefore (18) becomes

$$\begin{array}{ccc} \tilde{\gamma}^{k} \oplus \varepsilon & \longrightarrow & \tilde{\gamma}^{k+1} \\ \Big| & & \Big| \\ BSO(k) & \longrightarrow & BSO(k+1) \end{array}$$

and we get a cofibration

$$S(M\tilde{\gamma}^{k}) \to M(\tilde{\gamma}^{k+1}) \to M(\tilde{\gamma}^{k+1} \oplus \tilde{\gamma}^{k+1})$$

Applying the Pontrjagin-Thom construction to the exact sequence

$$\cdots \to \pi^{S}_{n+k}(M\tilde{\gamma}^{k}) \to \pi^{S}_{n+k+1}(M\tilde{\gamma}^{k+1}) \to \pi^{S}_{n+k+1}(M(\tilde{\gamma}^{k+1} \oplus \tilde{\gamma}^{k+1})) \to \cdots$$

establishes sequence (17).

The mapping $I\Omega_{n,k+1} \xrightarrow{e} \Omega^{\Delta SO}_{n-k-1}$ can be described as follows. Given an immersion $M^n \looparrowright \mathbb{R}^{n+k+1}$ consider M embedded in the normal bundle of the immersion ν_M via the zero section. Choosen a section $S : M \to \nu_M$ which is transverse to M. Then $N = S(M) \cap M$ is a submanifold of M whose normal bundle is given by

$$\nu_N = \nu_M|_N \oplus \nu_{S(M)}|_N = \nu_M|_N \oplus \nu_M|_N.$$

N with this ΔSO-structure represents the image of

$[M \hookrightarrow \mathbb{R}^{n+k+1}]$.

§5. SOME AUXILIARY COMPUTATIONS

We now make use of the computational techniques of [K, §9] to calculate $\Omega_i^{\xi k}$, $\Omega_i^{\Delta SO}$ for $0 \leq i \leq 2$.

Let $\phi = \phi^+ - \phi^-$ be a virtual vector bundle over a space X. By $\Omega_n(X;\phi)$ we will denote the n-th normal bordism group of X with coefficients in ϕ and $\overline{\Omega}_n(X;\phi)$ will denote the group of bordism classes of triplets (M,g, or) where M is a closed n-manifold, $g: M \to X$ is a map and or $\xi_M \cong \xi_{g^*\phi}$ is an isomorphism of the associated orientation bundles $\xi_M = \Lambda\tau_M$ and $\xi_{g^*\phi} = (\Lambda g^*\phi^+) \otimes (\Lambda g^*\phi^-)$.

See [K] for details.

Given a linear map $\ell : H^i(y;\mathbb{Z}_2) \to H^j(Z;\mathbb{Z}_2)$ between cohomology groups of finite $\mathbb{Z}_2$-dimensions, the adjoint of ℓ is the unique linear map

$$\ell_\wedge : H_j:(Z;\mathbb{Z}_2) \to H_i(y;\mathbb{Z}_2)$$

satisfying $\langle \ell(y), z\rangle = \langle y, \ell_*(z)\rangle$ for all $y \in H^i(Y;\mathbb{Z}_2)$, $z \in H_j(Z;\mathbb{Z}_2)$

Let the symbol μ stand for Hurewicz homomorphism from bordism to homology.

Let γ^k denote the k-dimensional vector bundle over $BO(k)$ and let σ denote the virtual bundle $(\gamma^2 \oplus \xi_{\gamma^2})\oplus\xi_{\gamma^2} - \gamma^2$.

Theorem 19 <u>Let</u> X <u>be a connected CW-complex with compact skeletons in all dimensions and let</u> $\phi = \phi^+ - \phi^-$ <u>be a virtual vector bundle over</u> X. <u>Then there is a commutative diagram of horizontal and vertical exact sequences.</u>

$$\begin{tikzcd}
 & \Omega_1^{fr}\,\Omega_1^{fr} \arrow[d] & & {[N,g,or]} \arrow[rr, mapsto] & & g^*w_2\phi & & & \\
\bar{\Omega}_3(X;\phi) \arrow[r, "i_3"] \arrow[d, "\mu"'] & \Omega_1(X\times BO(2);\phi+\sigma) \arrow[r, "\delta_2"] \arrow[d, "proj_*\circ\mu"] & \Omega_2(X;\phi) \arrow[r, "f_2"] & \bar{\Omega}_2(X;\phi) \arrow[r, "i_2"] & \mathbb{Z}_2 \arrow[r, "\delta_1"] \arrow[d, phantom, "\cong" rotate=90] & \Omega_1(X;\phi) \arrow[r, "f_1"] & \bar{\Omega}_1(X;\phi) \arrow[r] & 0 \\
H_3(X;\mathbb{Z}_2) \arrow[r, "(w_2\phi\cdot -)_\cap"'] & H_1(X;\mathbb{Z}_2) \arrow[d] & & & \Omega_1^{fr} & & & \\
 & 0 & & & & & &
\end{tikzcd}$$

<u>Moreover,</u> $\Omega_0(X;\phi) \cong \bar{\Omega}_0(X;\phi) \cong \begin{cases} \mathbb{Z} & \text{if } w_1\phi = 0 \\ \mathbb{Z}_2 & \text{if } w_1\phi \neq 0. \end{cases}$

δ_i, f_i <u>are the obvious forgetful homomorphisms. Finally,</u> δ_2 <u>is injective if</u> $w_2\phi = 0$, <u>and</u> $proj_*\circ\mu$ <u>is an isomorphism if and only if</u> $w_2\phi \neq 0$.

satisfying $\langle \ell(y), z\rangle = \langle y, \ell_*(z)\rangle$ for all $y \in H^i(Y;\mathbb{Z}_2)$, $z \in H_j(Z;\mathbb{Z}_2)$

Let the symbol μ stand for Hurewicz homomorphism from bordism to homology.

Let γ^k denote the k-dimensional vector bundle over $BO(k)$ and let σ denote the virtual bundle $(\gamma^2 \otimes \xi_{\gamma^2}) \oplus \xi_{\gamma^2} - \gamma^2$.

Theorem 19 <u>Let</u> X <u>be a connected CW-complex with compact skeletons in all dimensions and let</u> $\phi = \phi^+ - \phi^-$ <u>be a virtual vector bundle over</u> X. <u>Then there is a commutative diagram of horizontal and vertical exact sequences.</u>

$$\begin{tikzcd}
& \Omega_1^{fr}\Omega_1^{fr} \arrow[d] & & {[N,g,or]} \arrow[r, mapsto] & g^*w_2\phi & & & \\
\bar{\Omega}_3(X;\phi) \arrow[r, "i_3"] \arrow[d, "\mu"'] & \Omega_1(X\times BO(1);\phi+\sigma) \arrow[r, "\delta_2"] \arrow[d, "\mathrm{proj}_*\circ\mu"] & \Omega_2(X;\phi) \arrow[r, "f_2"] & \bar{\Omega}_2(X;\phi) \arrow[r, "j_2"] & \mathbb{Z}_2 \arrow[r, "\delta_1"] \arrow[d, phantom, "\cong" rotate=90] & \Omega_1(X;\phi) \arrow[r, "f_1"] & \bar{\Omega}_1(X;\phi) \arrow[r] & 0 \\
H_3(X;\mathbb{Z}_2) \arrow[r, "(w_2\phi\cdot -)_\wedge"'] & H_1(X;\mathbb{Z}_2) \arrow[d] & & & \Omega_1^{fr} & & & \\
& 0 & & & & & &
\end{tikzcd}$$

<u>Moreover,</u> $\Omega_0(X;\phi) \cong \bar{\Omega}_0(X;\phi) \cong \begin{cases} \mathbb{Z} & \text{if } w_1\phi = 0 \\ \mathbb{Z}_2 & \text{if } w_1\phi \neq 0. \end{cases}$

δ_i, f_i <u>are the obvious forgetful homomorphisms.</u> <u>Finally,</u> δ_2 <u>is injective if</u> $w_2\phi = 0$, <u>and</u> $\mathrm{proj}_*\circ\mu$ <u>is an isomorphism if and only if</u> $w_2\phi \neq 0$.

This is part of a theorem due to Koschorke and C. Olk. See [K, theorem 9.3].

If $x \in \mathbb{R}^k$ then $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix} = -\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix}$.

Hence ζ_k splits as $(\tilde{\gamma}^k \otimes \gamma^1) \oplus \tilde{\gamma}^k$ and $\Omega_i^{\zeta_k}$ can be written as the normal bordism group $\Omega_i(BO(1) \times BSO(k); (\gamma^1 \otimes \tilde{\gamma}^k) \oplus \tilde{\gamma}^k)$.

Since $\Lambda^{2k}(\tilde{\gamma}^k \oplus (\gamma^1 \otimes \tilde{\gamma}^k)) \cong \Lambda^k \tilde{\gamma}^k \otimes \Lambda^k (\tilde{\gamma}^1 \otimes \gamma^k) \cong \underbrace{\gamma^1 \otimes \ldots \otimes \gamma^1}_{k\text{-times}} \cong \begin{cases} \varepsilon & , k \text{ even} \\ \gamma^1 & , k \text{ odd} \end{cases}$

it follows that

$$(20) \quad \bar{\Omega}_i^{\zeta_k} = \bar{\Omega}_i(BO(1) \times BSO(k); (\gamma^1 \otimes \tilde{\gamma}^k) \oplus \gamma^k) \cong \begin{cases} \Omega_j(BO(1) \times BSO(k)) & , k \text{ even} \\ N_j(BSO(k)) & , k \text{ odd} \end{cases}$$

Note that for k odd the inverse isomorphism $N_j(BSO(k)) \to \bar{\Omega}_i^{\zeta_k}$ is given by

$$(21) \quad [N, \gamma] \to [N, (\varepsilon_{N,\gamma}), \text{ suitable isomorphism }]$$

Lemma 22 <u>The low dimensional bordism groups</u>

$\bar{\Omega}_j^{\zeta_k} = \bar{\Omega}_j(BO(1) \times BSO(k); \tilde{\gamma}^k \oplus \gamma^1 \otimes \tilde{\gamma}^k)$ <u>are isomorphic to</u>

	$j = 0$	$j = 1$	$j = 2$
k even > 2	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
k odd	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Proof. For k even it follows from (20) that

$\bar{\Omega}_1^{\zeta_k} \cong \Omega_1(BO(1)\times BSO(k)) \cong H_1(BO(1);\mathbb{Z}) \cong \mathbb{Z}_2$ generated by

$[S^1, (\text{Mobius} = M,*)]$ and

$\bar{\Omega}_2^{\zeta} \cong \Omega_2(BO(1)\times BSO(k)) \cong H_2(BO(1)\times BSO(k);\mathbb{Z}) \cong \Pi_2(BSO(k))$

$(k > 2)$ since $H_2(BO(1)\times BSO(k);\mathbb{Z}) \cong H_0(BO(1);\mathbb{Z}) \oplus H_2(BSO(k);\mathbb{Z})$.

The exact sequence

$$\begin{array}{ccccccc} H_2(BSO(k);\mathbb{Z}) & \xrightarrow{2} & H_2(BSO(k);\mathbb{Z}) & \to & H_2(BSO(k);\mathbb{Z}_2) & \to & H_1(BSO(k);\mathbb{Z}) \\ \wr\Vert & & \wr\Vert & & & & \Vert \\ \mathbb{Z}_2 & & \mathbb{Z}_2 & & & & 0 \end{array}$$

shows $H_2(BSO(k);\mathbb{Z}_2) \cong \Pi_2(BSO(k)) \cong \mathbb{Z}_2$ and the result follows from the isomorphism $N_2(X) \cong N_2 \oplus H_2(X;\mathbb{Z}_2)$. □

Lemma 23 $w_2(\zeta_k) = \begin{cases} 0 & k \equiv 0,1 \quad (4) \\ (w_1\gamma^1)^2 & k \equiv 2,3 \quad (4) \end{cases}$

Proof. We have a pull-back diagram of vector bundles

$$\begin{tikzcd} k\gamma^1 \arrow[r] \arrow[d, no head] & \gamma^1\otimes\tilde{\gamma}^k \arrow[d, no head] & \tilde{\gamma}^k \arrow[l] \arrow[d, no head] \\ BO(1) \arrow[r] & BO(1)\times BSO(k) & BSO(k) \arrow[l] \end{tikzcd}$$

and an isomorphism

$$H^2(BO(1)\times BSO(k);\mathbb{Z}_2) \cong H^2(BO(1);\mathbb{Z}_2)\oplus H^2(BSO(k);\mathbb{Z}_2).$$

It follows that $w_2(\gamma^1\otimes\tilde{\gamma}^k) = w_2(k\gamma^1) + w_2\tilde{\gamma}^k$ and therefore $w_2(\gamma^1\otimes\tilde{\gamma}^k \oplus\tilde{\gamma}^k) = w_2(k\gamma^1) = \binom{k}{2}(w_1\gamma^1)^2$ □

The inclusion $j : \mathbb{R}P^2 \to BO(1)\times BSO(n)$ induces isomorphism on the fundamental groups $(BO(1) = \mathbb{R}P^\infty)$ if $n > 2$. Therefore

$$j_*: \Omega_1(\mathbb{R}P^2; n\lambda) \to \Omega_1(BO(1)\times BSO(n); (\gamma^1\otimes\tilde{\gamma}^n)\oplus\tilde{\gamma}^n)$$

is an epimorphism. (λ denotes the canonical line bundle over $\mathbb{R}P^2$). Moreover, by comparing the corresponding sequences given by (19) it is easy to see that j_* is actually an isomorphism. The Pontrjagin-Thom construction and the fact that normal bordism groups depend only on the stability class in KO of the coefficients imply that

$$(24)\quad \Omega_1^{\zeta n} \cong \begin{cases} \Omega_2^S(\mathbb{R}P^3) & n \equiv 1 \quad (4) \\ \Omega_3^S(\mathbb{R}P^4/\mathbb{R}P^1) & n \equiv 2 \quad (4) \quad n > 2 \\ \Omega_4^S(\mathbb{R}P^5/\mathbb{R}P^2) & n \equiv 3 \quad (4) \\ \Omega_1^S(\mathbb{R}P^2 \sqcup *) & n \equiv 0 \quad (4) \end{cases}$$

Theorem 25. <u>The low dimensional normal bordism groups</u>

$\Omega_i(BO(1)\times BSO(n);(\gamma^1\otimes\bar{\gamma}^n)\oplus\bar{\gamma}^n) \cong \Omega_i^{\zeta n}$ <u>are given by the table</u>

	$\Omega_0^{\zeta n}$	$\Omega_1^{\zeta n}$	$\Omega_2^{\zeta n}$
$n \equiv 1$ (4)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8$
$n \equiv 2$ (4) $n > 2$	$\mathbb{Z}$	$\mathbb{Z}_4$	$\mathbb{Z}_2$
$n \equiv 3$ (4)	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$n \equiv 0$ (4)	$\mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Proof. The result for the 0-dimensional groups follows immediatly form (19). This is also the case for $\Omega_1(BO(1)\times BSO(n);(\gamma^1\otimes\bar{\gamma}^n)\oplus\bar{\gamma}^n)$, $n \equiv 3$ (4), since the sequence of theorem (19) becomes (See (20),(21))

$$
\begin{tikzcd}
\Omega_2(BSO(n)) \arrow[r] \arrow[d, "\cong"'] & \mathbb{Z}_2 \arrow[r] & \Omega_1(BO(1)\times BSO(n);(\gamma^1\otimes\tilde{\gamma}^n)\oplus\tilde{\gamma}^n) \arrow[r] & 0 \\
\Omega_2\oplus H_2(BSO(n)) \arrow[r] & \Omega_2 \arrow[u, "\cong"'] & &
\end{tikzcd} \tag{26}
$$

short exact sequence

$$
\begin{array}{ccccccc}
0 \to \Pi_3^S \mathbb{R}P^1 & \longrightarrow & \Pi_3^S \mathbb{R}P^4 & \longrightarrow & \Pi_3^S(\mathbb{R}P^4/\mathbb{R}P^1) & \to 0 \\
\wr\Vert & & \wr\Vert & & & \\
\mathbb{Z}_2 & & \mathbb{Z}_8 & & &
\end{array}
$$

showing $\mathbb{Z}_4 \cong \Pi_3^S(\mathbb{R}P^4/\mathbb{R}P^1) \cong \Omega_1^{\zeta n}$, $n \equiv 2\ (4)$, $n > 2$.

Next we compute the groups in dimension 2.

If $n \equiv 0\ (4)$ and $[S^1, f, \cong]\ \ \Omega_1(BO(1)\times BSO(n)\times BO(2); \tilde{\gamma}^n\oplus(\gamma^1\otimes\tilde{\gamma}^n)+\sigma))$

then the composite proj $\circ$ f : $S^1 \to BSO(n) \times BO(2)$ is null

homotopic showing $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \Omega_1(\mathbb{R}P^2; n\lambda)$

$\to \Omega_1(BO(1) \times BSO(n) \times BO(2); \tilde{\gamma}^n\oplus(\gamma^1\otimes\tilde{\gamma}^n)+\sigma)$ is onto. Theorem (19)

induces the following short exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_2 \arrow[r] & \Omega_2^{\zeta n} \arrow[r] & \Omega_2(BSO(n)) \arrow[r] \arrow[l, bend right, "J"'] & 0
\end{tikzcd}
$$

Let $\gamma : S^2 \to BSO(n)$ represent a generator of $\Omega_2(BSO(n)) \cong$ $\cong \Pi_2(BSO(n))$. We can define $r : \Omega_2(BSO(n)) \to \Omega_2(BO(1)\times BSO(n);\zeta_n)$ by $r[S^2,\gamma] = [S^2,(*,\gamma),\text{suitable isomorphism}]$ since $\tau_{S^2}\oplus\gamma\oplus\gamma$ is stable trivial ($\widetilde{KO}(S^2) = \mathbb{Z}_2$). Moreover, $2r[S^2,\gamma] = [S^2,*,\text{ some framing}] = 0$ since any framed 2-sphere is a framed boundary. Thus

$$\Omega_2^{\zeta}n \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for} \quad n \equiv 0 \ (4).$$

If $n \equiv 1(4)$ and $[S^1,f,=] \in \Omega_1(BO(1)\times BSO(n)\times BO(2);(\gamma^1\otimes\tilde{\gamma}^n)\oplus\tilde{\gamma}^n+\sigma)$ then either f is null-homotopic or there is a homotopy commutative diagram

$$\begin{tikzcd}
S^1 \arrow[r, "f"] \arrow[d, dash, "M"'] & BO(1)\times BSO(n)\times BO(2) \\
\mathbb{R}P^2 \arrow[r, "\Delta"] & \mathbb{R}P^2\times\mathbb{R}P^2 \arrow[r] & BO(1)\times BO(2) \arrow[ul]
\end{tikzcd}$$

where $[M]$ = generator of $\Pi_1(\mathbb{R}P^2)$ The pull-back diagram

$$\begin{tikzcd}
((n+2)\lambda\oplus\varepsilon^{n+1} - \lambda\oplus\varepsilon) \arrow[rrr] \arrow[d] & & & (\gamma^1\otimes\gamma^n)\oplus\tilde{\gamma}^n\oplus(\gamma^2\otimes\xi_{\gamma^2})\oplus\xi_{\gamma^2} \arrow[d] \\
\mathbb{R}P^2 \arrow[r] & \mathbb{R}P^2\times\mathbb{R}P^2 \arrow[r] & BO(1)\times BO(2) \arrow[r] & BO(1)\times BSO(n)\times BO(2)
\end{tikzcd}$$

shows there is an epimorphism

$$\Omega_1(\mathbb{R}P^2;2\lambda) \to \Omega_1(BO(1)\times BSO(n)\times BO(2);(\gamma^1\otimes\tilde{\gamma}^n)\oplus\tilde{\gamma}^n+\sigma)$$

and (19) induces a short exact sequence

$$0 \to \mathbb{Z}_4 \to \Omega_2(BO(1)\times BSO(n);\tilde{\gamma}^n\oplus(\gamma^1\otimes\tilde{\gamma}^n)) \to N_2 \oplus H_2(BSO(n);\mathbb{Z}_2) \to 0$$

The $H_2(BSO(n);\mathbb{Z}_2)\cong\Gamma_2(BSO(n))$ factor splits as in $n \equiv 0(4)$. Hence $\Omega_2(BO(1)\times BSO(n);\tilde{\gamma}^n\oplus(\gamma^1\otimes\tilde{\gamma}^n))\cong\Omega_2(BO(1);n\gamma^1)\oplus H_2(BSO(n);\mathbb{Z}_2)$. The result follows as $\Omega_2(BO(1);n\lambda)\cong\Omega_2(\mathbb{R}P^3;\lambda)\cong\mathbb{Z}_8$ $(KO(\mathbb{R}P^3)\cong\mathbb{Z}_4)$.

Assume now $n \equiv 2$ (4), $n > 2$. Theorem (19) gives a commutative diagram

$$\begin{tikzcd}
H_3(BO(1)\times BSO(n);\mathbb{Z}) \arrow[r, phantom, "\cong"] & \Omega_3(BO(1)\times BSO(n)) \arrow[r, "j_3"] \arrow[d] & \Omega_1(BO(1)\times BSO(n)\times BO(2);(\gamma^1\otimes\tilde{\gamma}^n)\oplus\tilde{\gamma}^n+\sigma) \arrow[d, "\cong(19)"] \\
H_3(BO(1);\mathbb{Z}) \arrow[u] \arrow[dr, "\cong"'] & H_3(BO(1)\times BSO(n);\mathbb{Z}_2) \arrow[r] \arrow[d] & H_1(BO(1)\times BSO(n);\mathbb{Z}_2) \\
 & H_3(BO(1);\mathbb{Z}_2) \arrow[r, "\cong"', "\text{adj to } (w,\gamma^1)^2"'] & H_1(BO(1);\mathbb{Z}_2) \arrow[u, "\cong"']
\end{tikzcd}$$

showing j_3 is onto and (19) becomes

$$0 \to \Omega_2(BO(1)\times BSO(n);(\gamma^1\otimes\tilde{\gamma}^n)\oplus\tilde{\gamma}^n) \to \underset{\displaystyle \mathbb{Z}_2}{\underset{\wr\shortparallel}{\Omega_2(BO(1)\times BSO(n))}} \to 0 \quad (22)$$

Finally we consider the case $n \equiv 3$ (4). We have then the following commutative diagram

$$
(21)\qquad
\begin{tikzcd}
{[N,\gamma]} & \mathcal{N}_3(BSO(n)) \arrow[r, "j_3"] \arrow[d] & \Omega_1(BO(1)\times BSO(n)\times BO(2);\ (\gamma^1\otimes\bar{\gamma}^n)\oplus\bar{\gamma}^n+\sigma) \arrow[d, "\cong\ (19)"] \\
 & H_3(BO(1)\times BSO(n);\mathbb{Z}_2) \arrow[r] \arrow[d] & H_1(BO(1)\times BSO(n);\mathbb{Z}_2) \arrow[d, "\cong"] \\
\xi_N[N] \arrow[uu, mapsto] & H_3(BO(1);\mathbb{Z}_2) \arrow[r, "\text{adj to}"', "(w,\gamma^1)^2"] & H_1(BO(1);\mathbb{Z}_2)
\end{tikzcd}
$$

Thus j_3 factors through $\mathcal{N}_3 = 0$. Theorem (19) induces a short exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \Omega_2(BO(1)\times BSO(n);(\gamma^1\otimes\bar{\gamma}^n)\oplus\bar{\gamma}^n) \arrow[r] & \operatorname{im} f_2 \arrow[r] \arrow[d, phantom, "\wr\|\,(26)"] & 0 \\
 & & & H_2(BSO(n);\mathbb{Z}_2) &
\end{tikzcd}
$$

which splits as in $n \equiv 0$ (4) $\square$

Let $\Omega_i^{\phi}n$ denote the unoriented bordism groups analogous to $\Omega_i^{r}n$. They were computed by Koschorke and Olk [K, § 10]. It will be useful to have a detailed evaluation of the forgetful homomorphism

$$\Omega_2^{\zeta} n \to \Omega_2^{\phi} n$$

If $n \equiv 0 \ (4)$ theorem (19) induces a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega_1^{fr} \oplus H_1(BO(1);\mathbb{Z}_2) \arrow[r] \arrow[d] & \Omega_2^{\zeta} n \arrow[r] \arrow[d] & \Omega_2(BSO(n)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_1(BO(1);\mathbb{Z}_2) \arrow[r] & \Omega_2^{\phi} n \arrow[r] & \Omega_2(BO(n)) \arrow[r] & 0
\end{tikzcd}$$

that proves

$$\begin{tikzcd}
\Omega_2^{\zeta} n \arrow[d] & \cong & \Omega_1^{fr} \oplus H_1(BO(1) \oplus \Omega_2(BSO(n)) \arrow[d] & \cong & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[d, "0\times 1\times 1"] \\
\Omega_2^{\phi} n & \cong & H_1(BO(1) \oplus \Omega_2(BO(n)) & \cong & \mathbb{Z}_2 \oplus \mathbb{Z}_2
\end{tikzcd} \tag{27}$$

If $n \equiv 1 \ (4)$, the corresponding diagram is

$$[N, (\lambda, \gamma), \hat{\xi}] \longmapsto (w_2\gamma, (w_1 N)^2)$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_4 \arrow[r] \arrow[d] & \Omega_2(BO(1)\times BSO(n); (\gamma^1 \otimes \hat{\gamma}^n) \oplus \hat{\gamma}^n) \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & \Omega_2(BO(1)\times BO(n); (\gamma^1 \otimes \gamma^n) \oplus \gamma^n) \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

$$[N, (\lambda, \gamma), \xi] \longmapsto (w_2\gamma, (w_1\gamma)^2, (w_1 N)^2)$$

Therefore $\Omega_2^{\zeta}n \to \Omega_2^{\phi}n$ is given by

$$(28) \qquad \mathbb{Z}_2 \oplus \mathbb{Z}_8 \xrightarrow{\ \psi\ } \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \qquad (n \equiv 1\ (4))$$

where $\psi(1,0)=(1,0,0)$ and $\psi(0,1)=(0,0,1)$.

For $n \equiv 3\ (4)$ theorem (19) induces the commutative diagram of short exact sequences

$$\begin{tikzcd}
 & {[N,(\lambda,\gamma),\cong]} \arrow[r, mapsto] & \omega_2\gamma & \\
0 \to H_1(BO(1);\mathbb{Z}_2) \arrow[r] \arrow[d] & \Omega_2(BO(1)\times BSO(n);(\gamma^1\oplus\tilde{\gamma}^n)\oplus\tilde{\gamma}^n) \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r, dash] & \Omega_2(BO(1)\times BO(n);(\gamma^1\oplus\gamma^n)\oplus\gamma^n) \arrow[r] & \mathbb{Z}_2\oplus\mathbb{Z}_4 \arrow[r] & 0 \\
 & {[N,(\lambda,\gamma),\cong]} \arrow[r, mapsto] & (\omega_2\gamma,(\omega_1\gamma)^2) &
\end{tikzcd}$$

Therefore we get

$$(29) \qquad \begin{array}{ccc} \Omega_2^{\zeta}n & \longrightarrow & \Omega_2^{\phi}n \\ \wr\Vert & & \wr\Vert \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \xrightarrow[1\,\times\,0]{} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array} \qquad (n \equiv 3\ (4))$$

A similar argument shows that

(30) $$\Omega_2^{\zeta}n \cong \Omega_2^{\phi}n \quad \text{for} \quad n \equiv 2\ (4)$$

Next we compute the bordism groups $\Omega_i^{\Delta SO}$ of the sequence (17) for $i = 0,1,2$.

Clearly we can identify $\Omega_i^{\Delta SO}$ with $\Omega_i(BSO(N);\tilde{\gamma}^N \oplus \tilde{\gamma}^N)$, N large, and since $w_1(2\tilde{\gamma}^N) = w_2(2\tilde{\gamma}^N) = 0$ by (19) we have a commutative diagram of exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Omega_1^{fr} \arrow[r, "\delta_2"] \arrow[d] & \Omega_2^{\Delta SO} \arrow[r] \arrow[d] & \Omega_2(BSO(N)) \arrow[r, "0"] \arrow[d, "\cong"] \arrow[l, dashed, bend right, "r"'] & \Omega_1^{fr} \arrow[r, "\delta_1"] \arrow[d, "id"] & \Omega_1^{\Delta SO} \arrow[r] \arrow[d] & 0 \arrow[d] & \\
& 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\cong"'] & \Omega_2(BO(N)) \arrow[r, "0"'] & \Omega_1^{fr} \arrow[r] & \mathbb{Z}_4 \arrow[r] & \Omega_1(BO(1)) \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is the unoriented sequence (See [Sal])

A splitting homomorphism $r : \Omega_2(BSO(n)) \to \Omega_2^{\Delta SO}$ is defined formally the same as in theorem (25). Hence we have

(31) $$\begin{tikzcd}
\Omega_2^{\Delta SO} \cong \Omega_1^{fr} \oplus \Omega_2(BSO(N)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[d] \arrow[d, shift left=8] \arrow[d, shift left=20, "\pi_2"] \\
\Omega_2^{\Delta O} \cong \Omega_2(BO(N)) \cong \mathbb{Z}_2
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\Omega_1^{\Delta SO} \cong \mathbb{Z}_2 \arrow[d] & \Upsilon \\
\Omega_1^{\Delta O} \cong \mathbb{Z}_4 .
\end{tikzcd}$$

§6 THE ORIENTED COBORDISM RING

In this section we recall some well know facts about the oriented cobordism ring. The main references are [St] and [Wa].

The structure of the unoriented cobordism ring N_* was determined by Thom. N_* is a ring of poynomials over $\mathbb{Z}_2$ with a generator x_i in each dimension not of the form 2^j-1. A necessary and sufficient condition that two manifolds be unoriented cobordant is that they have the same Stiefel-Whitney numbers. A manifold M^k $(k \neq 2^j-1)$ represents a generator iff the Stiefel-Whitney number $s_k[M^k] = 1$.

The analysis of the oriented cobordism ring Ω_* is due the Thom, Milnor and Wall. They proved

(32) The Ω_k are finitely generated abelian groups

(33) The torsion free part of Ω_* is a polynomial ring $\mathbb{Z}[h_4, h_8, h_{12}, \ldots]$ and a manifold M^{4k} qualifies as a generator iff the Pontrjagin number

$$\rho_k\,[M^{4k}] = \begin{cases} \pm\, p & \text{if } 2k+1 \text{ is a power of a prime } p \\ \pm\, 1 & \text{if } 2k+1 \text{ is not a prime power.} \end{cases}$$

(34) All torsion elements of Ω have order 2.

(35) $\left\{\begin{array}{l}\text{Two manifolds are (oriented) cobordant iff they} \\ \text{have the same Stiefel-Whitney and Pontrjagin numbers.}\end{array}\right.$

(36) $\left\{\begin{array}{l}\text{The image of the forgetful homomorphism } \Omega_* \to N_* \\ \text{consists precisely of those classes for which all} \\ \text{Stiefel-Whitney numbers with factor } w_1 \text{ vanish.}\end{array}\right.$

Representatives for generators h_{4k} can be given as follows. See [Mi, § 16]. Let m,n be integers satisfying $m + n = 2k+1$, $m,n \geq 2$. Let $H_{m,n}$ be a non-singular complex hypersurface of degree (1,1) in the product $\mathbb{C}P^m \times \mathbb{C}P^n$ of complex projective spaces. $H_{m,n}$ is a 4k-dimensional oriented manifold whose Pontrjagin number $\rho_k[H_{m,n}] = -\binom{m+n}{n} = -\frac{(m+n)!}{m!n!}$

Lemma 37. <u>Let</u> p <u>be a prime number such that</u> $p^r \mid m$ <u>but</u> $p^{r+1} \nmid m$. <u>Then</u> $p^r \,\Big|\, \binom{p^\alpha m}{p^\alpha}$ <u>and</u> $p^{r+1} \nmid \binom{p^\alpha m}{p^\alpha}$ □

(38) Now assume $2k+1 = q^{\alpha}$ for some prime q. If $\alpha = 1$ then $\rho_k[\mathbb{C}P^{2k}] = q$. If $\alpha > 1$, then by the previous lemma $q \mid \binom{q^{\alpha}}{q^{\alpha-1}}$ and $q^2 \nmid \binom{q^{\alpha}}{q^{\alpha-1}}$. This shows that the greatest common divisor g.c.d. $\left(q^{\alpha}, \binom{q^{\alpha}}{q^{\alpha-1}}\right) = q$ and therefore there are integers β_0, β_1 satisfying

$$\rho_k[\beta_0 \mathbb{C}P^{2k} + \beta_1 H_{q^{\alpha}-q^{\alpha-1}, q^{\alpha-1}}] = q.$$

If $2k+1 = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, $r > 1$, then by lemma (37) $p_i \nmid \binom{2k+1}{p_i^{\alpha_i}}$. Thus the g.c.d. $\left(2k+1, \binom{2k+1}{p_1^{\alpha_1}}, \ldots, \binom{2k+1}{p_r^{\alpha_r}}\right) = 1$ and there are integers $\beta_0, \ldots, \beta_r$ satisfying

$$\rho_k[\beta_0 \mathbb{C}P^{2k} + \beta_1 H_{2k+1-p_1^{\alpha_1}, p_1^{\alpha_1}} + \ldots + \beta_r H_{2k+1-p_r^{\alpha_r}, p_r^{\alpha_r}}] = 1.$$

Further, Wall [Wa] defined an irredundant set of generators for Ω_* as follows.

The Dold manifold $P(m,n)$ is the orbit space of the $\mathbb{Z}_2$-action $(x,z) \to (-x,\bar{z})$ on $S^m \times \mathbb{C}P^n$, where $\bar{z}$ denotes complex conjugates of the homogeneous coordinates $[z_0, \ldots, z_n]$. Let T reflect S^m in the plane $x_m = 0$. Then $(x,z) \to (Tx,z)$ is compatible with the $\mathbb{Z}_2$-action and defines a diffeomorphism A of $P(m,n)$. In the case m is odd and n even, $P(m,n)$ is orientable and A reverses the orientation. Let $Q(m,n)=P(m,n)\times I/(p,0)\sim(Ap,1)$. If k is not a power of 2 write $k=2^{r-1}(2s+1)$. Define a set of generators $x_i\in\mathcal{N}_*$ by

$$x_{2k-1}=[P(2^r-1,\ 2^rs)]$$

$$x_{2k}=\ [Q(2^r-1,2^rs)]$$

$$x_{2^j}=\ [\mathbb{R}P^{2^j}].$$

Let W_* denote the polynomial algebra generated by $\{x_{2k-1},x_{2k},(x_{2^j})^2\}_{k\text{ not a power of }2.}$ A derivation ∂ on W_* can be defined by $\partial x_{2k-1}=0$, $\partial x_{2k}=x_{2k-1}$, $\partial(x_{2^j})^2=0$ and induces the exact triangle

(39)
$$\begin{tikzcd}
\Omega_* \arrow[rr, "2"] & & \Omega_* \arrow[dl] \\
& W_* \arrow[ul, "\partial"] &
\end{tikzcd}$$

For each partition $w = (a_1, \ldots, a_r)$ with all a_i not a power of 2, $a_i \neq a_j$ for $i \neq j$, define an element g_w of Ω_* by $g_w = \partial(x_{2a_1} \cdots x_{2a_r})$. Then

(40) The g_w and h_{4k} form an irredundant set of generators.

§7 ORIENTED BORDISM GROUPS OF IMMERSIONS

Given a selftransverse immersion of an oriented n-manifold into $\mathbb{R}^{n+k}$ the normal bundle of the double points manifold has, in a natural way, a $\mathbb{Z}_2 \wr SO(k)$-structure. Moreover, there is a "double points homomorphism" $I\Omega_{n,k} \to \Omega_{n-k}^{\mathbb{Z}_2 \wr SO(k)}$, where $\Omega_{n-k}^{\mathbb{Z}_2 \wr SO(k)}$ denotes the bordism group of (n-k)-manifolds with a $\mathbb{Z}_2 \wr SO(k)$-structure on their stable normal bundle. (See remarks on page 33).

Theorem 41 <u>For $n > 0$, $I\Omega_{n,n} \cong \Omega_n \oplus \mathbb{Z}$ if n is even and $I\Omega_{n,n} \cong \Omega_n \oplus \mathbb{Z}_2$ if n is odd. The $\mathbb{Z}$ or $\mathbb{Z}_2$ factor is generated by the class of the immersion</u> $j_n : S^n \looparrowright \mathbb{R}^{2n}$ (See §4).

Proof. From Whitney immersion theorem and sequence 16) we get the commutative diagram with horizontal exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Omega_0^{\xi_n} \arrow[r] \arrow[dr, "\cong"'] & I\Omega_{n,n} \arrow[r] \arrow[d] & \Omega_n \arrow[r] & 0 \\
 & & \Omega_0^{\mathbb{Z}_2 \wr SO(n)} & &
\end{tikzcd}$$

□

<u>Remark</u> Even though $I\Omega_{1,1} \cong \Omega_1^{fr}$ does not lie in the metastable range the result is still valid.

We say that an oriented manifold M^n <u>immerses up to oriented cobordism into $\mathbb{R}^{n+k}$</u> if $[M]$ belongs to the image of the forgetful homomorphism $f: I\Omega_{n,k} \to \Omega_n$.

Let $IN_{n,k}$ denote unoriented bordism group of immersions of n-manifolds into $\mathbb{R}^{n+k}$. A theorem of R. Brown [B] shows that the forgetful homomorphism $IN_{n,n-2} \to N_n$ is onto only if n is not a power of 2. We also have

Theorem 42. <u>Every oriented n-manifold immerses up to oriented cobordism into</u> $\mathbb{R}^{2n-2}$ <u>if and only if n is not a power of 2</u>.

Proof. Consider the commutative diagram associated to sequence 16 §4

$$\begin{tikzcd}
I\Omega_{n,n-2} \arrow[r, "f"] \arrow[d] & \Omega_n \arrow[r] \arrow[d] & \Omega_1^{\xi}n-2 \arrow[r] \arrow[d] & I\Omega_{n-1,n-2} \arrow[r, "f"] & \Omega_{n-1} \arrow[r] & 0 \\
IN_{n,n-2} \arrow[r] & N_n \arrow[r] & \Omega_1^{\phi}n-2 & & &
\end{tikzcd}$$

If $\equiv 1(4)$ the result follows as $\Omega_1^{\xi}n-2 = 0$. This also shows

(43) $$I\Omega_{n,n-1} = \Omega_n \quad \text{for} \quad n \equiv 0(4)$$

If $n \equiv 3$ (4) then the homomorphisms $\Omega_n \to \Omega_n$ and $\Omega_1^{\xi\, n-2} \to \Omega_1^{\phi\, n-2}$ are both injective. The result follows from R. Brown's theorem [B].

The case $n \equiv 2$ (4) follows from (40)$^{\mathrm{fc}}$ since generators g_w and h_{4k} never have these dimensions.

Finally we consider the case $n = 4k$. It is sufficient to look at generators h_{4k}. Assume first $4k$ is not a power of 2. h_{4k} is represented by a sum of $\mathbb{C}P^{2k}$'s and $H_{m,\ell}$'s. $\mathbb{C}P^{2k}$ immerses in $\mathbb{R}^{8k-2}$. $H_{m,\ell} \subset \mathbb{C}P^m \times \mathbb{C}P^\ell$ with $m,\ell \geq 2$ and either m or ℓ odd. Hence by [D-M] $H_{m,\ell} \subset \mathbb{C}P^m \times \mathbb{C}P^\ell \looparrowright \mathbb{R}^{8k-2}$.

Now assume n is a power of 2. The exact sequence

$$
\begin{array}{ccccc}
I\Omega_{n,n-2} & \to & I\Omega_{n,n-1} & \xrightarrow{e} & \mathbb{Z}_2 \\
 & & \wr\!\shortparallel & & \\
 & & \Omega_n & &
\end{array}
\qquad (43)
$$

implies that any odd multiple of h_n immerses up to oriented cobordism into $\mathbb{R}^{2n-2}$ if and only if h_n does.

But by [L-M, thm. 4.1] $\mathbb{C}P^{n/2}$ does not immerse up to oriented cobordism into $\mathbb{R}^{2n-2}$. Notice that coker $f \cong \mathbb{Z}_2$. $\square$

Theorem 44. *For $n > 3$ the groups $I\Omega_{n,n-1}$ are given by*

$$I\Omega_{n,n-1} \cong \begin{cases} \Omega_n & n \equiv 0 \quad (4) \\ \Omega_n \oplus \mathbb{Z}_2 & n \equiv 2 \quad (4) \\ \Omega_n \oplus \mathbb{Z}_4 & n \equiv 3 \quad (4),\ \alpha(n+1) > 1 \\ \Omega_n \oplus \mathbb{Z}_2 & n \equiv 3 \quad (4),\ \alpha(n+1) = 1 \end{cases}$$

If $n \equiv 1$ (4) there is a short exact sequence

$$0 \to \mathbb{Z}_2 \to I\Omega_{n,n-1} \to \Omega_n \oplus \mathbb{Z}_2 \to 0$$

Proof. The case $n \equiv 0$ (4) was established in (43). Suppose now $n \equiv 2(4)$. By (41)and results of [K,§10], the diagram associated to $(16)^{14}$ becomes

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_2 & \to & I\Omega_{n,n-1} & \to & \Omega_n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbb{Z}_4 & \to & N_n \oplus \mathbb{Z}_4 & \to & N_n & \to & 0 \end{array}$$

(39)[16] implies $\Omega_n \to N_n$ is injective and so it is

$\mathbb{Z}_2 = \Omega_1^{\zeta}n-1 \to \Omega_1^{\phi}n-1 \cong \mathbb{Z}_4$. Therefore $I\Omega_{n,n-1}$ injects into

$IN_{n,n-1}$ and the top sequence splits.

If $n \equiv 1(4)$ theorem 25[15] and sequence 16[14] imply

$|I\Omega_{n,n-1}| = 4|\Omega_n|$. Therefore sequence 17 becomes

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow I\Omega_{n,n-1} \longrightarrow \Omega_n \oplus \mathbb{Z}_2 \longrightarrow 0$$

Finally assume $n \equiv 3$ (4), $\alpha(n+1) > 1$. The sequences 16[14], 17[14] take

the form

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathbb{Z}_2 \arrow[d] & & \\
0 \arrow[r] & \mathbb{Z}_4 \arrow[r, "\partial"] & I_{n,n-1} \arrow[r, "f"] \arrow[d, "g"] & \Omega_n \arrow[r] & 0 \\
 & & \Omega_n \oplus \mathbb{Z}_2 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

The vertical sequences shows $I\Omega_{n,n-1}$ does not have elements of order 8 and hence the horizontal sequence splits. If $\alpha(n+1)=1$ then (17)§4 looks like

$$I\Omega_{n+1,n} \xrightarrow{e} \mathbb{Z}_2 \xrightarrow{0} I\Omega_{n,n-1} \to \Omega_n \oplus \mathbb{Z}_2 \to 0$$

showing $I\Omega_{n,n-1} \cong \Omega_n \oplus \mathbb{Z}_2$. $\square$

We now study the homomorphism

$$f : I\Omega_{n,n-3} \to \Omega_n$$

Clearly f is onto if $n \equiv 2(4)$. In fact, (40)§6 and (42) show that $I\Omega_{n,n-4} \to \Omega_n$ is onto for $n \equiv 2\ (4)$.

If $n \equiv 1\ (4)$, associated to (16)§4 there is a commutative diagram with exact rows

$$\begin{tikzcd}
I\Omega_{n,n-3} \arrow[r, "f"] \arrow[d] & \Omega_n \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[d, "= (30)§5"] \\
IN_{n,n-3} \arrow[r] & N_n \arrow[r] & \mathbb{Z}_2
\end{tikzcd}$$

that shows f is onto whenever $IN_{n,n-3} \to N_n$ is onto.

This is the case unless $n - 1$ is a power of 2. If $n = 2^m + 1$ Koschorke [K, pages 122,123] exhibited an oriented manifold M^n satisfying $\bar{w}_2\,\bar{w}_{n-2}(M)\,[M]=1$. See also (55) §8.

If $n \equiv 3\ (4)$ then generators g_w in this dimension are given by partitions $w = (a_1,\dots,a_r)$ of $\frac{n+1}{2}$ with all a_i not a power of 2 and unequal. $g_w = \partial x_{2a_1} \dots x_{2a_r}$ (See (40) §6).

Let $\alpha(n)$ denote the number of ones in the binary expansion of n. Since $\alpha(2a_i) \geq 2$ [B] implies that $x_{2a_1} \dots x_{2a_r}$ has a representative that immerses into $\mathbb{R}^{2n-2}$ if $r \geq 2$ or $\alpha(a_1) \geq 4$. As g_w is "trivially embedded" into $x_{2a_1} \dots x_{2a_r}$ we have that $g_w \in \operatorname{im} f$ unless $w = (2^m + 2^\ell)$ or $w = (2^m + 2^\ell + 2^j)$, $m > \ell > j > 1$.

The following lemma proves $x_{2^{m+1}+2^{\ell+1}+2^{j+1}}$ has also a representative that immerses into $\mathbb{R}^{2n-2}$.

Hence if $n \equiv 3(4)$ and $\alpha(n+1) \neq 2$ f is onto.

Lemma 45. $Q(2^j-1, 2^{m-1}+2^{\ell-1})$ <u>immerses up to unoriented bordism in</u> $\mathbb{R}^{2n-2}$, <u>where</u> $n + 1 = \dim Q$ <u>and</u> $m > \ell > 1$.

Proof. $H^*(Q;\mathbb{Z}_2)$ is generated by x,c,d in dimensions 1,1,2 respectively, with the relations,

$$x^2 = 0,\ c^{2^j} = c^{2^j-1}\cdot x,\ d^{2^{m-1}+2^{\ell-1}+1} = 0.$$

The Stiefel-Whitney class of Q is

$$w(Q)=(1+c+x)\ (1+c)^{2^j-2}\ (1+c+d)^{2^{m-1}+2^{\ell-1}+1}$$

See [Wa] for details.

$H^{n-2}(Q;\mathbb{Z}_2)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$ and is generated by $c^{2^j-3}\,d^{2^{m-1}+2^{\ell-1}}$ and $c^{2^j-1}\,d^{2^{m-1}+2^{\ell-1}-1}$

$$\begin{aligned}\bar{w}(Q) &= (1+c+x)^{-1}\left[(1+c)^{2^j-2}\right]^{-1}\left[(1+c+d)^{2^{m-1}+2^{\ell-1}+1}\right]^{-1}\\ &= (1+c+c^2+\ldots+c^{2^j}+x+xc^2+\ldots+xc^{2^j})(1+c^2+c^{2^j})(1+c+d)^{2^{m-1}-2^{\ell-1}-1}\\ &= (1+c+x+c^{2^j})\ (1+c+d)^{2^{m-1}-2^{\ell-1}-1}\\ &= (1+c+x+c^{2^j})\ (\Sigma\, c^i d^j) \text{ where } j \leq 2^{m-1}-2^{\ell-1}-1\end{aligned}$$

and hence $\bar{w}_{n-2}(Q) = 0$.

By Theorem 6.3 of [B] Q is unoriented cobordant to a manifold that immerses in $\mathbb{R}^{2n-2}$ $\square$

We now consider the case $n \equiv 0(4), \alpha(n)>1$. We have a commutative diagram with exact row.

$$\begin{tikzcd} I\Omega_{n,n-3} \arrow[r, "g"] \arrow[dr, "f"'] & I\Omega_{n,n-2} \arrow[r, "e"] \arrow[d] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ & \Omega_n & \end{tikzcd} \quad (42)$$

This shows that if $[M] \in \Omega_n$ then $2[M] \in \operatorname{im} f$. Therefore the elements that might not belong to im f are of the form mh_{4k} + decomposables, m odd. (33)(c) now implies

Lemma 46. Let $\alpha(4k) > 2$. The homomorphism $f: \Omega_{4k,4k-3} \to \Omega_{4k}$ is onto iff there exists an oriented manifold M^{4k} such that $[M] \in \operatorname{im} f$ and the Pontrjagin number $p_k[M]$ is odd. □

If $\alpha(4k) \geq 3$ then $p_k[\mathbb{C}P^{2k}] = 2k+1$ and $\mathbb{C}P^{2k} \rightarrowtail \mathbb{R}^{4k-3}$ by [D-M]. Therefore f is onto for $\alpha(4k) \geq 3$.

If $\alpha(4k)=2$ then $\mathbb{C}P^{2k}$ does not immerse up to oriented cobordism into $\mathbb{R}^{8k-3}$ [L-M, thm. 4.1] and coker $f \cong \mathbb{Z}_2$.

Finally we study the case $4k = 2^m$. It is well known that $[\mathbb{C}P^{2k}] = [\mathbb{R}P^{2k} \times \mathbb{R}P^{2k}] \in N_{4k}$. The Stiefel-Whitney number $\bar{w}_2 \cdot \bar{w}_{4k-2}(\mathbb{C}P^{2k})[\mathbb{C}P^{2k}] = \bar{w}_2 \cdot \bar{w}_{4k-2}(\mathbb{R}P^{2k} \times \mathbb{R}P^{2k})[\mathbb{R}P^{2k} \times \mathbb{R}P^{2k}]$ is non-zero. The commutative diagram

$$\begin{array}{ccccc} I\Omega_{4k,4k-3} & \xrightarrow{f} & \Omega_{4k} & \xrightarrow{S} & \mathbb{Z}_2 \oplus \mathbb{Z}_8 \\ \downarrow & & \downarrow & & \downarrow \Psi \quad (28)\S 5 \\ IN_{4k,4k-3} & \to & N_{4k} & \to & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}$$

$$[\mathbb{R}P^{2k} \times \mathbb{R}P^{2k}] \mapsto (1,0,0)$$

shows that $S(h_{4k}) = (1,2a) \in \mathbb{Z}_2 \oplus \mathbb{Z}_8$ and therefore coker $f \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$.

We summarize all these results in the following

Theorem 47. Let $n > 5$. The forgetful homomorphism

$$f : I\Omega_{n,n-3} \to \Omega_n$$

is onto if and only if either

i) $n \equiv 2$ (4),

ii) $n \equiv 1$ (4) and $\alpha(n) > 2$, or

iii) $n \equiv 0$ (4) and $\alpha(n) > 2$.

If $n \equiv 3(4)$ and $\alpha(n+1) \neq 2$ then f is onto. If $n+1 = 2^m + 2^\ell$, $m > \ell > 1$, then f is onto if and only if the Dold manifold $P(2^\ell - 1, 2^{m-1})$ immerses up to oriented cobordism in $\mathbb{R}^{2n-3}$. □

<u>Remark</u>. Ralph Cohen has announced the truth of the "immersion conjecture". It states that any m-manifold immerses in $\mathbb{R}^{2n-\alpha(n)}$. This implies that $P(2^{\ell}-1, 2^{m-1})$ immerses into $\mathbb{R}^{2n-3}$ ($n = \dim P = 2^m + 2^{\ell} - 1$) and hence $f: I\Omega_{n,n-3} \to \Omega_n$ is onto for $n \equiv 3$ (4). However no proof has yet appeared.

Next we study the groups $I\Omega_{n,n-2}$. Let $n \equiv 0$ (4), $n > 4$. If $\alpha(n) \geq 2$, then $I\Omega_{n+1,n-1} \to \Omega_{n+1}$ is onto (47) and we have the commutative diagrams (See (30)§5 and [K,§10])

$$\alpha(n) \geq 3 \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, "\cong"] & I\Omega_{n,n-2} \arrow[r] \arrow[d] & \Omega_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & N_n \oplus \mathbb{Z}_2 \arrow[r] & N_n \arrow[r] & 0 \end{tikzcd}$$

$$\alpha(n) = 2 \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, "\cong"] & I\Omega_{n,n-2} \arrow[r] \arrow[d] & \Omega_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "\partial"] & N_n/\mathbb{Z}_2 \oplus \mathbb{Z}_4 \arrow[r] & N_n \arrow[r] & 0 \end{tikzcd}$$

The upper diagram shows $I\Omega_{n,n-2} \cong \Omega_n \oplus \mathbb{Z}_2$. The $\mathbb{Z}_4$-factor in the lower diagram is generated by an immersion of $\mathbb{R}P^{2^m} \times \mathbb{R}P^{2^{\ell}}$ into $\mathbb{R}^{2n-2}$, $n = 2^m + 2^{\ell}$, $m > \ell$ say, and

$\mathbb{Z}_2 \xrightarrow{\partial} IN_{n,n-2}$ maps the generator to two times this immersion.

If $a \in H^1(\mathbb{R}P^{2^m};\mathbb{Z}_2)$, $b \in H^1(\mathbb{R}P^{2^\ell};\mathbb{Z}_2)$ are the generators then the total Stiefel-Whitney class of $\mathbb{R}P^{2^m} \times \mathbb{R}P^{2^\ell}$ is given by

$$\begin{aligned} w(\mathbb{R}P^{2^m} \times \mathbb{R}P^{2^\ell}) &= (1+a)^{2^m+1} \times (1+b)^{2^\ell+1} \\ &= (1+a+a^{2^m})\ (1+b+b^{2^\ell}) \\ &= 1+(a+b)+ab+b^{2^\ell}+ab^{2^\ell}+a^{2^m}+a^{2^m}b + a^{2^m}b^{2^\ell} \end{aligned}$$

and $w_1(\mathbb{R}P^{2^m}\times\mathbb{R}P^{2^\ell})^{2^m} \cdot w_{2^\ell}(\mathbb{R}P^{2^m}\times \mathbb{R}P^{2^\ell})\ [\mathbb{R}P^{2^m}\times \mathbb{R}P^{2^\ell}] =$

$$= (a^{2^m}+b^{2^m})b^{2^\ell}[\mathbb{R}P^{2^m}\times\mathbb{R}P^{2^\ell}] = a^{2^m}b^{2^\ell}[\mathbb{R}P^{2^m}\times\mathbb{R}P^{2^\ell}] = 1.$$

By (36)§4 $\mathbb{R}P^{2^m} \times \mathbb{R}P^{2^\ell}$ is not cobordant to an oriented manifold. Therefore the top sequence of the lower diagram also splits.

If $\alpha(n)=1$ then sequence (16)§4 becomes (See (42) and (47))

$$\mathbb{Z}_2 \xrightarrow{0} I\Omega_{n,n-2} \xrightarrow{f} \Omega_n \xrightarrow{[M] \mapsto \rho_{n/4}[M]} \mathbb{Z}_2 \to 0$$

showing $I\Omega_{n,n-2}$ is isomorphic to the subgroup of Ω_n consisting of classes $[M]$ with $\rho_{n/4}[M]$ even.

Now assume $n \equiv 3\ (4)$. If $\alpha(n+1) \geq 3$ then we have a commutative diagram (See (47),(28)35 and [K,§10])

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_8 \arrow[r] \arrow[d, "\psi"] & I\Omega_{n,n-2} \arrow[r] \arrow[d] & \Omega_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & N_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & N_n \arrow[r] & 0
\end{tikzcd}$$

where $\psi(1,0)=(1,0,0)$ and $\psi(0,1)=(0,0,1)$. This implies that

$$I\Omega_{n,2n-2} \cong \Omega_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8 .$$

If $\alpha(n+1)=2$ then we have the exact sequences

$$\begin{tikzcd}
I\Omega_{n+1,n-2} \arrow[r, "f"] & \Omega_{n+1} \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_8 \arrow[r] & I\Omega_{n,n-2} \arrow[r] & \Omega_n \arrow[r] & 0 \\
I\Omega_{n+1,n-1} \arrow[r, "e"] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & I\Omega_{n,n-2} \arrow[r] & I\Omega_{n,n-1} \arrow[r] \arrow[d, phantom, "\cong\ (44)" description] & 0 & \\
& & & \Omega_n \oplus \mathbb{Z}_4 & &
\end{tikzcd}$$

By (47) coker $f \cong \mathbb{Z}_2$ and thus $|I\Omega_{n,n-2}|=8|\Omega_n|$. Hence image $e \cong \mathbb{Z}_2$ and $I\Omega_{n,n-2}$ is an extension of $\Omega_n \oplus \mathbb{Z}_4$ by $\mathbb{Z}_2$.

If $\alpha(n+1) = 1$, $n+1 = 4k$ say, then we get the diagram

$$\begin{tikzcd}
 & & & \mathbb{Z}_2\oplus\mathbb{Z}_2 \arrow[d] \\
I\Omega_{n+1,n-2} \arrow[r, "f"] & \Omega_{n+1} \arrow[r, "{[\alpha P^k]\mapsto(1,2\alpha)}"] & \mathbb{Z}_2 \oplus \mathbb{Z}_8 \arrow[r] & I\Omega_{n,n-2} \arrow[d] \\
 & & & I\Omega_{n,n-1} \cong \Omega_n \oplus \mathbb{Z}_2 \ (44) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

The vertical sequence shows $I\Omega_{n,n-2}$ has no elements of order 8 and thérefore coker $f \cong \mathbb{Z}_4$ and $I\Omega_{n,n-2} \cong \Omega_n \oplus \mathbb{Z}_4$.

If $n \equiv 2(4)$, $\alpha(n+2) \neq 2$ we have a commutative diagram (See (27) and [K, §10])

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[d, "\text{projection}"] & I\Omega_{n,n-2} \arrow[r] \arrow[d] & \Omega_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & IN_{n,n-2} \arrow[r] & N_n \arrow[r] & 0
\end{tikzcd}$$

showing $I\Omega_{n,n-2}$ is an extension of $\Omega_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by $\mathbb{Z}_2$.

Finally let $n \equiv 1$ (4). Then $I\Omega_{n+1,n-2} \xrightarrow{f} \Omega_{n+1}$ is onto and we get a short exact sequence

(48) $\quad 0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to I\Omega_{n,n-2} \to \Omega_n \to 0$

and for $\alpha(n-1) > 1$ a commutative diagram with exact rows

$$\begin{tikzcd}
I\Omega_{n+1,n-1} \arrow[r, "e"] \arrow[d] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r, "\delta"] \arrow[d, "\pi_2"] & I\Omega_{n,n-2} \arrow[r, "g"] \arrow[d] & I\Omega_{n,n-1} \arrow[r, "e"] \arrow[d] & \mathbb{Z}_2 \arrow[r] \arrow[d] & \Omega_{n-1} \arrow[d] \\
IN_{n+1,n-1} \arrow[r] & \mathbb{Z}_2 \arrow[r] & N_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & N_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_4 \arrow[r] & \Omega_{n-1} \oplus \mathbb{Z}_2
\end{tikzcd}$$

where $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\pi_2} \mathbb{Z}_2$ is the projection onto the second factor and $\mathbb{Z}_2 \to \mathbb{Z}_4$ is the monomorphism. See (31)§5,(47), (25)§5. Theorems 41 and 44 imply that $I\Omega_{n,n-1} \xrightarrow{e} \mathbb{Z}_2$ is onto. A diagram chasing argument shows that the image of $N_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ is the subgroup of order two, $IN_{n+1,n-1} \to \mathbb{Z}_2$ is the zero homomorphism and image of $I\Omega_{n+1,n-1} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is $\mathbb{Z}_2 \oplus 0$.

Thus the diagram above reduces to a diagram with short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, "\cong"] & I\Omega_{n,n-2} \arrow[r] \arrow[d] & \mathrm{img} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & N_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & N_n \oplus \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

and therefore $I\Omega_{n,n-2} \cong \text{img} \oplus \mathbb{Z}_2$, where $\text{img} \subset I\Omega_{n,n-1}$ has index two.

If $\alpha(n-1) = 1$ then (48) shows that $|I\Omega_{n,n-2}| = 4|\Omega_n|$ and sequence 17 becomes

$$0 \to \mathbb{Z}_2 \to I\Omega_{n,n-2} \xrightarrow{g} I\Omega_{n,n-1} \to \mathbb{Z}_2 \to 0$$

If ν_j denotes the normal bundle associated to the immersion $j: S^{n-1} \looparrowright \mathbb{R}^{2n-2}$, then the following equality holds [L-S, corollary 3.2]

$$\langle e(\nu_j), [S^{n-1}] \rangle = 2$$

where $e(\nu_j)$ is the Euler class of ν_j. This induces a commutative diagram of exact sequences (See (44))

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \Omega_1^{fr} \arrow[d, "\partial"'] \arrow[dr, "0"] & & \\
I\Omega_{n,n-2} \arrow[r, "g"] & I\Omega_{n,n-1} \arrow[r] \arrow[d] & \mathbb{Z}_2 \arrow[r] & 0 \\
 & \Omega_n \oplus \partial(\Omega_1(BO(1))) \arrow[ur, "\pi"'] \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

where Π factors as $\Omega_n \otimes \partial\Omega_1(BO(1)) \to \partial\Omega_1(BO(1)) \cong \mathbb{Z}_2$.

Hence all elements of $\operatorname{im} g$ have order two and $\operatorname{im} g \cong \Omega_n \otimes \mathbb{Z}_2$.

Thus we have

Theorem 49. i) <u>If $n \equiv 0\ (4)$ then</u> $I\Omega_{n,n-2} \cong \Omega_n$ <u>if $\alpha(n)>1$.</u>

<u>If n is a power of 2 then</u> $I\Omega_{n,n-2}$ <u>is isomorphic to the subgroup of</u> Ω_n <u>consisting of classes $[M]$ with Pontrjagin number</u> $p_{n/4}[M]$ <u>even.</u>

ii) <u>If $n \equiv 1(4)$ then there is a short exact sequence</u>

$$0 \to \mathbb{Z}_2 \to I\Omega_{n,n-2} \to \Omega_n \otimes \mathbb{Z}_2 \to 0$$

<u>This sequence splits if $\alpha(n) > 2$.</u>

iii) <u>If $n \equiv 2(4)$ and $\alpha(n+2) \neq 2$ then</u> $I\Omega_{n,n-2}$ <u>is an extension of</u> $\Omega_n \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$ <u>by</u> $\mathbb{Z}_2$.

iv) <u>If $n \equiv 3(4)$ then</u> $I\Omega_{n,n-2} \cong \begin{cases} \Omega_n \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_8, & \alpha(n+1) \geq 3 \\ \Omega_n \otimes \mathbb{Z}_4, & \alpha(n+1) = 1. \end{cases}$

<u>If $\alpha(n+1)=2$ then</u> $I\Omega_{n,n-2}$ <u>is an extension of</u> $\Omega_n \otimes \mathbb{Z}_4$ <u>by</u> $\mathbb{Z}_2$. $\square$

<u>Remark</u> Under the assumption of the "immersion conjecture" the condition $\alpha(n+2) \neq 2$ can be dropped.

§8 DOUBLE POINTS AND EMBEDDINGS.

The monomorphism $\mathbb{Z}_2 \times SO(k) \to \mathbb{Z}_2 \wr SO(k)$ induces a homomorphism of the bordism groups $\Omega^{\zeta}_{1}k \to \Omega_1^{\mathbb{Z}_2 \wr SO(k)}$. Moreover, there is a commutative diagram

$$
\begin{tikzcd}
\Omega^{\zeta}_{1}k \arrow[rr, "\partial"] \arrow[dr] & & I\Omega_{k+1,k} \arrow[dl] \\
& \Omega_1^{\mathbb{Z}_2 \wr SO(k)} &
\end{tikzcd}
$$

where $I\Omega_{k+1,k} \to \Omega_1^{\mathbb{Z}_2 \wr SO(k)}$ is the double points homomorphism.

The homotopy sequences of the fibrations

$$E(\mathbb{Z}_2 \times SO(k)) \to B(\mathbb{Z}_2 \times SO(k))$$

$$E(\mathbb{Z}_2 \wr SO(k)) \to B(\mathbb{Z}_2 \wr SO(k))$$

induce a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \pi_1(B\mathbb{Z}_2 \wr SO(k)) \arrow[r] & \pi_0(\mathbb{Z}_2 \wr SO(k)) \arrow[r] & 0 \\
0 \arrow[r] & \pi_1(B(\mathbb{Z}_2 \times SO(k))) \arrow[r] \arrow[u, "\therefore \cong"] & \pi_0(\mathbb{Z}_2 \times SO(k)) \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}
$$

showing $B(\mathbb{Z}_2 \times SO(k)) \to B(\mathbb{Z}_2 \wr SO(k))$ is a 1-equivalence and therefore $\Omega_1^{\xi_k} \to \Omega_1^{\mathbb{Z}_2 \wr SO(k)}$ is onto. This implies $\Omega_1^{\mathbb{Z}_2 \wr SO(k)} \cong 0$ for $k \equiv 3(4)$.

Lemma 50 <u>Let</u> X,Y <u>be CW complexes</u>, ξ^k <u>a vector bundle over the wedge</u> $X \vee Y$. <u>Then there exists an exact sequence</u>

$$\ldots \to \Omega_n^{fr} \to \Omega_n(X;\xi|_X) \oplus \Omega_n(Y;\xi|_Y) \to \Omega_n(X \vee Y;\xi) \to \Omega_{n-1}^{fr} \to \ldots$$

Proof. $(M\xi; M(\xi|_X), M(\xi|_Y))$ is an excisive triad. Hence we have the Mayer-Vietoris sequence

$$\to \pi_{n+k}^s(S^k) \to \pi_{n+k}^s(M(\xi|_X)) \oplus \pi_{n+k}^s(M(\xi|_Y)) \to \pi_n^s(M\xi) \to \ldots$$

that gives the required sequence under the Pontrjagin-Thom isomorphism. $\square$

$B(\mathbb{Z}_2 \wr SO(k)) \simeq S^\infty \times_{\mathbb{Z}_2} B(SO(k))^2$ whose 2-skeleton is $\mathbb{R}P^2 \vee \mathbb{C}P^1 \vee \mathbb{C}P^1$. The universal $\mathbb{Z}_2 \wr SO(k)$ vector bundle restricts to $k\lambda \oplus \varepsilon^k$ over $\mathbb{R}P^2$ and $\gamma^2 \oplus \varepsilon^{2k-2}$ over each copy of $\mathbb{C}P^1$. Since $\Omega_1(\mathbb{C}P^1;\gamma^2) \cong \Omega_1^{SO(2)} = 0$ the exact sequence of the lemma becomes

$$\Omega_1^{fr} \to \Omega_1(\mathbb{R}P^2;k\lambda) \to \Omega_1^{\mathbb{Z}_2 \wr SO(k)} \to \Omega_0^{fr} \cong \mathbb{Z}$$

But we have seen that $\Omega_1(\mathbb{R}P^2;k\lambda) \cong \Omega_1^{\xi_k}$ and that for $k \not\equiv 3\ (4)$ Ω_1^{fr} injects into $\Omega_1^{\xi_k}$. Theorem 25 implies then that $\Omega_1^{\mathbb{Z}_2 \wr SO(k)} \cong \mathbb{Z}_2$ if k is even and $\Omega_1^{\mathbb{Z}_2 \wr SO(k)} = 0$ for k odd.

Moreover, for k even there is a commutative diagram

(51)
$$\begin{tikzcd}
0 \arrow[r] & \Omega_1^{fr} \arrow[r] & \Omega_1^{\xi_k} \arrow[r] \arrow[d] & \Omega_1^{\mathbb{Z}_2 \wr SO(k)} \arrow[r] & 0 \\
 & & I\Omega_{k+1,k} \arrow[ur, "D"'] \arrow[d] & & \\
 & & \Omega_{k+1} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where D stands for the double points homomorphism.

We now compute the groups $\Omega_2^{\mathbb{Z}_2 \wr SO(k)}$. As normal bordism groups can be written as $\Omega_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; S^\infty \times_{\mathbb{Z}_2} \gamma^2)$, where γ^2 denotes the product bundle of the k-dimensional universal oriented vector bundle $\gamma \to BSO(k)$ with itself.

We will need to calculate the low dimensional homology groups of $S^\infty \times_{\mathbb{Z}_2} BSO(k)^2$ with $\mathbb{Z}_2$ and $\mathbb{Z}$ coefficients. Consider the following commutative diagram of fibrations.

$$\begin{tikzcd}
BSO(k)^2 \arrow[r] & S^\infty \times_{\mathbb{Z}_2} BSO(k)^2 \arrow[r] & \mathbb{R}P^\infty \\
BSO(k) \arrow[r] \arrow[u, "\Delta"] & \mathbb{R}P^\infty \times BSO(k) \arrow[r, dashed] \arrow[u, "I^2"] & \mathbb{R}P^\infty \arrow[u, "id"]
\end{tikzcd}$$

where Δ denotes the diagonal.

Let τ be the involution of $BSO(k) \times BSO(k)$ which transposes coordinates. Let K^* and I^* denote, respectively, the kernel and image of

$$id + \tau^*: H^*(BSO(k)^2; \mathbb{Z}_2) \to H^*(BSO(k)^2; \mathbb{Z}_2)$$

K^* and I^* are graded groups with $I^* \subset K^*$. Using the projection $S^\infty \times_{\mathbb{Z}_2} BSO(k)^2 \to \mathbb{R}P^\infty$ we regard $H^*(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2)$ as an $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2)$-module. The mod-2 cohomology of $S^\infty \times_{\mathbb{Z}_2} BSO(k)^2$ can be computed as follows. See [Th] for details.

Theorem 52 (Steenrod). <u>There is an isomorphism of $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)$-modules</u>

$$H^*(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}_2) \cong (H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) \otimes K^*/I^*) \oplus I^*$$

<u>where</u> $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)$ <u>acts trivially on</u> I^*. $\square$

We have then the following table of the mod 2 cohomology of $BSO(k)$, $BSO(k)^2$, $S^\infty \times_{\mathbb{Z}_2} BSO(k)^2$, $\mathbb{R}P^\infty \times BSO(k)$.

	H^0	H^1	H^2	H^3
$BSO(k)$	1	0	w_2	w_3
$BSO(k)^2$	1	0	$1\otimes w_2, w_2\otimes 1$	$1\otimes w_3, w_3\otimes 1$
K	1	0	$1\otimes w_2 + w_2\otimes 1$	$1\otimes w_3 + w_3\otimes 1$
I	0	0	$1\otimes w_2 + w_2\otimes 1$	$1\otimes w_3 + w_3\otimes 1$
$S^\infty \times_{\mathbb{Z}_2} BSO(k)^2$	1	x	u_2, x^2	u_3, x^3
$\mathbb{R}P^\infty \times BSO(k)$	1	x	w_2, x^2	$w_3, x^3, w_2 \cdot x$

where $j^* u_i = 1\otimes w_i + w_i \otimes 1$ and $x \in H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$ is the generator. Note that $u_i \cdot x = 0$ as $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2)$ acts trivially on I^*. Since the induced bundles $I^*(S^\infty \times_{\mathbb{Z}_2} \gamma^2)$ and

$j^*(S^\infty \times_{\mathbb{Z}_2} \gamma^2)$ are isomorphic to γ^2 and ζ_k, respectively,

it is easy to see that

$$w_1(S^\infty \times_{\mathbb{Z}_2} \gamma^2) = \begin{cases} x & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

and

$$w_2(S^\infty \times_{\mathbb{Z}_2} \gamma^2) = \begin{cases} u_2 & k \equiv 0,1\ (4) \\ u_2 + x^2 & k \equiv 2,3\ (4) \end{cases}$$

Recall that the 2-skeleton of $S^\infty \times_{\mathbb{Z}_2} BSO(k)^2 = \mathbb{R}P^2 \vee \mathbb{C}P^1 \vee \mathbb{C}P^1$

The commutative diagram

$$\begin{tikzcd} H_2(\mathbb{C}P^1 \vee \mathbb{C}P^1;\mathbb{Z}) \arrow[r, two heads] \arrow[d, "\cong"] & H_2(BSO(k)^2;\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[d] \\ H_2(\mathbb{R}P^2 \vee \mathbb{C}P^1 \vee \mathbb{C}P^1;\mathbb{Z}) \arrow[r, two heads] & H_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}) \end{tikzcd}$$

shows $H_2(BSO(k)^2;\mathbb{Z}) \to H_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z})$ is an epimorphism

and therefore induces a commutative diagram

$$\begin{array}{ccccccc}
H_2(BSO(k)^2;\mathbb{Z}) & \xrightarrow{0} & H_2(BSO(k)^2;\mathbb{Z}) & & & & \\
\downarrow & & \downarrow & & & & \\
H_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}) & \xrightarrow[\cdot 2]{} & H_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}) \to & H_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}_2) \to & H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}) \to 0 \\
& & & \wr\| & \wr\| \\
& & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2
\end{array}$$

which shows $H_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}) \cong \mathbb{Z}_2$.

Hence, if k is even $\overline{\Omega}_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; S^\infty \times_{\mathbb{Z}_2} \gamma^2) \cong$ $\cong \Omega_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2) \cong \mathbb{Z}_2$.

If k is odd $\overline{\Omega}_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; S^\infty \times_{\mathbb{Z}_2} \gamma^2)$ is the bordism group of triplets $[M,g\ or]$ where the composite

$$M \xrightarrow[g]{} S^\infty \times_{\mathbb{Z}_2} BSO(k)^2 \longrightarrow \mathbb{R}P^\infty$$

classifies the orientation bundle ξ_M of M. The obvious homomorphism

$$\overline{\Omega}_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; S^\infty \times_{\mathbb{Z}_2} \gamma^2) \to N_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2) \cong N_2 \oplus I_2 \oplus H_2(\mathbb{R}P^\infty;\mathbb{Z}_2)$$

is then given by $[M,g,or] \longmapsto ([M], \alpha u_2^*, \xi_M[M])$ for some $\alpha \in \mathbb{Z}_2$. Here I_2 denotes the vector space generated by the dual of $u_2 \in H^2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2)$. Therefore for k odd

$$\bar{\Omega}_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; S^\infty \times_{\mathbb{Z}_2} \gamma^2) \cong N_2 \oplus I_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 .$$

We now apply theorem 19 §5 to calculate $\Omega_2^{\mathbb{Z}_2 \wr SO(k)}$.

Note first that $(w_2 \cdot -)_\wedge : H_3(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2) \to H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2)$ is zero if $k \equiv 0,1\ (4)$ and onto if $k \equiv 2,3\ (4)$.

So if $k \equiv 0,1\ (4)$ then $H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2) \to \Omega_2^{\mathbb{Z}_2 \wr SO(k)}$ is a monomorphism. The following commutative diagram shows this is also the case for $k \equiv 3\ (4)$

$$\begin{array}{ccccc}
[N,g,or] & & \bar{\Omega}_3(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; S^\infty \times_{\mathbb{Z}_2} \gamma^2) & \xrightarrow{j_3} & H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2) \\
\downarrow & & \downarrow & & \downarrow \cong \\
\zeta_N[N] & & H_3(\mathbb{R}P^\infty; \mathbb{Z}_2) & \xrightarrow[(x^2 \cdot -)_\wedge]{\cong} & H_1(\mathbb{R}P^\infty; \mathbb{Z}_2)
\end{array}$$

since j_3 factors through $N_3 = 0$.

However, if $k \equiv 2\ (4)$ by (25) §6 we have a commutative diagram

$$\begin{array}{ccc}
H_3(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}) & \xrightarrow{j_3} & H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2) \\
\uparrow & & \uparrow \cong \\
H_3(\mathbb{R}P^\infty \times BSO(k); \mathbb{Z}) & \xrightarrow[j_3]{} & H_1(\mathbb{R}P^\infty \times BSO(k); \mathbb{Z}_2)
\end{array}$$

that implies $H_3(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}) \to H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2)$ is onto.

Hence, if k is even the sequence given by (19) §5 becomes

$$H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2) \to \Omega_2^{\mathbb{Z}_2 \wr SO(k)} \to H_2(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}_2 \to 0$$

showing

$$\Omega_2^{\mathbb{Z}_2 \wr SO(k)} \cong \begin{cases} \mathbb{Z}_2 & k \equiv 0 \ (4) \\ 0 & k \equiv 2 \ (4) \end{cases}$$

If $k \equiv 1$ (4) we get a commutative diagram of exact sequences (See (25) §5).

$$\begin{tikzcd}
 & 0 & & & & \\
0 \arrow[r] & H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2; \mathbb{Z}_2) \arrow[u] \arrow[r] & \Omega_2^{\mathbb{Z}_2 \wr SO(k)} \arrow[r] & n_2 \oplus \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}_4 \arrow[u] \arrow[r] & \mathbb{Z}_8 \oplus \mathbb{Z}_2 \arrow[u] \arrow[r] & N_2 \oplus H_2(BSO(k); \mathbb{Z}_2) \arrow[u, "id \oplus 0"'] \arrow[r] & 0 \arrow[u, leftrightarrow] & \\
 & \Omega_1^{fr} \arrow[u] & & & & \\
 & 0 \arrow[u] & & & &
\end{tikzcd}$$

where the bottom sequence corresponds to $\Omega_2^{r}k$. This implies

$\Omega_2^{\mathbb{Z}_2 \wr SO(k)} \cong \mathbb{Z}_4$.

For $k \equiv 3\ (4)$ the associated diagram becomes

$$\begin{tikzcd}
 & & & \mathcal{N}_2 \arrow[dr, "\cong"] & & \\
0 \arrow[r] & H_1(S^\infty \times_{\mathbb{Z}_2} BSO(k)^2;\mathbb{Z}_2) \arrow[r] & \Omega_2^{\mathbb{Z}_2 \wr SO(k)} \arrow[r] & \mathcal{N}_2 \oplus I_2 \arrow[u] \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \\
0 \arrow[r] & H_1(\mathbb{R}P^\infty \times BSO(k);\mathbb{Z}_2) \arrow[r] \arrow[u, "\cong"] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[u] & \mathcal{N}_2 \oplus H_2(BSO(k);\mathbb{Z}_2) \arrow[r] \arrow[u, "id \times 0"] \arrow[d] & \mathbb{Z}_2 \arrow[r] \arrow[u, "id"] & 0 \\
 & & & \mathcal{N}_2 \arrow[ur, "\cong"] & &
\end{tikzcd}$$

showing $\Omega_2^{\mathbb{Z}_2 \wr SO(k)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We have then the following table.

Theorem 53 <u>The bordism groups</u> $\Omega_i^{\mathbb{Z}_2 \wr SO(k)}$ <u>are given by the following table</u> $(0 \leq i \leq 2)$

	$\Omega_0^{\mathbb{Z}_2 \wr SO(k)}$	$\Omega_1^{\mathbb{Z}_2 \wr SO(k)}$	$\Omega_2^{\mathbb{Z}_2 \wr SO(k)}$
$k \equiv 0\ (4)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$k \equiv 1\ (4)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_4$
$k \equiv 2\ (4)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
$k \equiv 3\ (4)$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

□

It is easy to see that the natural homomorphism

$$\Omega_2^{\zeta_k} \longrightarrow \Omega_2^{\mathbb{Z}_2 \wr SO(k)}$$

fits in the following exact sequences

$$0 \to H_2(BSO(k);\mathbb{Z}_2) \otimes \Omega_1^{fr} \to \Omega_2^{\zeta}k \to \Omega_2^{\mathbb{Z}_2 \wr SO(k)} \to 0 \quad k \equiv 0,1(4)$$

$$0 \to H_2(BSO(k);\mathbb{Z}_2) \to \Omega_2^{\zeta}k \to \Omega_2^{\mathbb{Z}_2 \wr SO(k)} \to I_2 \longrightarrow 0 \quad k \equiv 3\ (4)$$

We now consider the problem of representing classes of immersions by embeddings. We have the exact sequence $(n < 2k-1)$

$$(54) \quad \cdots \to E\Omega_{n,k} \to I\Omega_{n,k} \xrightarrow{D} \Omega_{n-k}^{\mathbb{Z}_2 \wr SO(k)} \to E\Omega_{n-1,k} \to \cdots$$

due to Salomorsen [Sal], where $E\Omega_{n,k}$ denotes the bordism group of classes of embeddings of oriented n-manifolds into $\mathbb{R}^{n+k}$ and D stands for the double points homomorphism.

If n is odd, $n > 1$, we obtain the short exact sequence

$$0 \to E\Omega_{n,n} \to \Omega_n \oplus \mathbb{Z}_2 \xrightarrow{D} \mathbb{Z}_2 \to 0$$

and if n is even, $n > 2$, we obtain

$$\mathbb{Z}_2 \xrightarrow{0} E\Omega_{n,n} \to \Omega_n \oplus \mathbb{Z} \xrightarrow{D} \mathbb{Z} \to 0 \tag{51}$$

For $k = n-1, n-2$ the following diagram of exact sequences are obtained (See (44)[§7], (49)[§7], (51). All homology groups are mod 2 coefficients)

$n \equiv 2(4)$

$$\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \arrow[d] & & \\
 & H_2(BSO(n-2)) \otimes \Omega_1^{fr} \arrow[d] \arrow[dr, "0"] & & & \Omega_1^{fr} \arrow[d] \arrow[dr, "0"] & & \\
E\Omega_{n,n-2} \arrow[r] & I\Omega_{n,n-2} \arrow[d] \arrow[r, "D"'] & \mathbb{Z}_2 \arrow[r] & E\Omega_{n-1,n-2} \arrow[r, "0"] & I\Omega_{n-1,n-2} \arrow[d] \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \\
 & \Omega_n \oplus H_1(\mathbb{R}P^\infty) \arrow[d] \arrow[r, "\Pi"'] & H_1(\mathbb{R}P^\infty) \arrow[u, "\cong"] & & \Omega_{n-1} \oplus H_1(\mathbb{R}P^\infty) \arrow[d] \arrow[r] & H_1(\mathbb{R}P^\infty) \arrow[u, "\cong"'] & \\
 & 0 & & & 0 & &
\end{tikzcd}$$

$n \equiv 0(4)$

$$\begin{tikzcd}
 & & & & \Omega_1^{fr} \arrow[d] \arrow[dr, "0"] & & \\
E\Omega_{n,n-2} \arrow[r] & I\Omega_{n,n-2} \arrow[r] & 0 \arrow[r] & E\Omega_{n-1,n-2} \arrow[r] & I\Omega_{n-1,n-2} \arrow[d] \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \\
 & & & & \Omega_{n-1} \oplus H_1(\mathbb{R}P^\infty) \arrow[d] \arrow[r] & H_1(\mathbb{R}P^\infty) \arrow[u, "\cong"'] & \\
 & & & & 0 & &
\end{tikzcd}$$

$n \equiv 3 \ (4)$

$$\begin{tikzcd}
 & \mathbb{Z}_2 \oplus \mathbb{Z}_8 \arrow[rr, "Proj"] \arrow[d] & & \mathbb{Z}_8 \arrow[d] & & \\
E\Omega_{n,n-2} \arrow[r] & I\Omega_{n,n-2} \arrow[rr, "D"] \arrow[d] & & \mathbb{Z}_4 \arrow[r, "0"] \arrow[d] & E\Omega_{n-1,n-2} \arrow[r] & \Omega_{n-1} \oplus \mathbb{Z}_2 \arrow[r, "D"] & 0 \\
 & \Omega_n \arrow[urr, "0"] \arrow[d] & & 0 & & \\
 & 0 & & & &
\end{tikzcd}$$

$n \equiv 1 \ (4)$

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & H_2(BSO(n-2)) \oplus H_1(\mathbb{R}P^\infty) \arrow[rr, "Proj"] \arrow[d] & & H_1(\mathbb{R}P^\infty) \arrow[d] & & \\
E\Omega_{n,n-2} \arrow[r] & I\Omega_{n,n-2} \arrow[rr, "D"] \arrow[d, "f"'] & & H_1(\mathbb{R}P^\infty) \oplus I_2 \arrow[r] & E\Omega_{n-1,n-2} \arrow[r] & \Omega_{n-1} \arrow[r, "D"] & 0 \\
 & \Omega_n \arrow[urr, "\psi"'] \arrow[d] & & & & \\
 & 0 & & & &
\end{tikzcd}$$

We need to evaluate ψ above in order to have a complete description of the double points homomorphims. By exactness of (54) it is equivalent to study the problem of embedding an oriented n-manifold up to oriented cobordism in $\mathbb{R}^{2n-2}$, $n \equiv 1 \ (4)$. Every orientable n-manifold embeds in $\mathbb{R}^{2n-1}$ [M-P], thus we need only to investigate whether generators of Ω_* in dimensions n embed up to oriented cobordism in $\mathbb{R}^{2n-2}$, $n \equiv 1 \ (4)$.

Let $w = (a_1, \ldots, a_r)$ be a partition of $(n+1)/2$ with all

a_i not a power of 2 and $a_i \neq a_j$ for $i \neq j$. Generators in dimension n are given by $g_w = \partial(x_{2a_1} \cdots x_{2a_r})$. See (39) and (40). Each class $x_{2a} \in N_{2a}$ has a representative that immerses into $\mathbb{R}^{4a-\alpha(a)}$ and embeds in $\mathbb{R}^{4a-\alpha(a)+1}$ [B].

By [B, lemma 2.1], $x_{2a_1} \cdots x_{2a_r}$ has a representative that embeds in $\mathbb{R}^{2n+3-\Sigma\alpha(a_i)}$, where $2a_1 + \ldots + 2a_r = n+1$. Thus for $r \geq 2$ g_w embeds in $\mathbb{R}^{2n-2}$ unless $r = 2$, $\alpha(a_1) = \alpha(a_2) = 2$.

If $n = 4S + 1$ then ∂x_{n+1} is represented by the Dold manifold $P(1,2S)$.

Proposition 55. <u>Let $n = 4S + 1$. $P(1,2S)$ embeds up to oriented cobordism in $\mathbb{R}^{2n-2}$ if and only if $n-1$ is not a power of 2.</u>

We give the proof at the end of the section.

If $w = a_1 + a_2$, $a_1 = 2^{m+\ell} + 2^\ell$, $a_2 = 2^t + 1$, $m,\ell,t \geq 1$ then x_{2a_1}, x_{2a_2} are represented by $Q_1 = Q(2^{\ell+1} - 1, 2^{m+\ell})$, $Q_2 = Q(1,2^t)$. Let P_1,P_2 denote the Dold manifolds $P(2^{\ell+1} - 1, 2^{m+\ell})$, $P(1,2^t)$ respectively and let V

$V = P_1 \times P_2 \times I/(p_1,p_2,0) \sim (Ap_1,Ap_2,1)$. The projection $(p_1,p_2,t) \mapsto t$ induces a fibre map $\beta: V \to S^1$ with fibre

$P_1 \times P_2$ which fits into the pull-back diagram

(56)
$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & Q_1 \times Q_2 \arrow[d, "\beta_1 \times \beta_2"] \\ S^1 \arrow[r, hook] & S^1 \times S^1 \\ Z \arrow[r, mapsto] & (Z, Z^{-1}) \end{tikzcd}$$

implying $\partial(x_{2a_1} \cdot x_{2a_2}) = [V] \in \Omega_{2a_1+2a_2-1}$ (See [Wa]).

Proposition 57. Let $n = 2a_1 + 2a_2 - 1$ and V^n be as above. Then V^n embeds in $\mathbb{R}^{2n-2}$.

We give the proof at the end of the section.

Propositions 55,57 and the previous remarks show that if $n \equiv 1\ (4)$ then every oriented n-manifold embeds up to oriented cobordism in $\mathbb{R}^{2n-2}$ if and only if $n - 1$ is not a power of 2. Thus the homomorphism $\psi : \Omega_n \to \Omega_2^{\mathbb{Z}_2 \wr SO(1)}$ is zero if and only if $\alpha(n) > 2$. The diagrams on page 99 now imply

Theorem 58 For $n > 1$, $E\Omega_{n,n} \cong \Omega_n$. For $n > 3$,

$$E\Omega_{n,n-1} = \begin{cases} \Omega_n \oplus \mathbb{Z}_2 & \underline{\text{If either } n\equiv 1,2\ (4),\ \text{or } n\equiv 3\ (4) \text{ and } \alpha(n+1)>1.} \\ \Omega_n & \underline{\text{If either } n\equiv 3(4) \text{ and } \alpha(n+1)=1, \text{or } n\equiv 0(4) \text{ and } \alpha(n)=1.} \end{cases}$$

<u>If $n \equiv 0(4)$ and $\alpha(n)>1$ there is a short exact sequence</u>

$$0 \to \mathbb{Z}_2 \to E\Omega_{n,n-1} \to \Omega_n \to 0 \quad \square$$

Theorem 59 <u>Every oriented n-manifold embeds up to oriented cobordism in $\mathbb{R}^{2n-2}$ if and only if one of the following conditions holds</u>

i) $n \equiv 2,3\ (4)$.

ii) $n \equiv 1\ (4)$, $\alpha(n) > 2$.

iii) $n \equiv 0\ (4)$, $\alpha(n) > 1$. $\square$

<u>Proof of proposition 55</u>.

The mod 2 cohomology of $P(1,2s)$ was determined by Dold [D] who showed that is the ring with two generators c,d in dimension 1,2 respectively, with the relations $c^2 = d^{2s+1} = 0$.

He also proved that the Stiefel-Whitney class of $P(1,2s)$ is given by

$$w(P) = (1+c)\,(1+c+d)^{2s+1}$$

In particular $w_2(P) = d$ and $w_3(P) = cd$.

The normal class is then

$$\begin{aligned}\bar{w}(P) &= (1+c)\,\Sigma\binom{2s+i}{i}(c+d)^i \\ &= (1+c)\,\Sigma\binom{2s+i}{i}(d^i+icd^{i-1}) \\ &= \Sigma\binom{2s+i}{i}(d^i+icd^{i-1}+cd^i)\end{aligned}$$

$H^{n-2}(P;\mathbb{Z}_2)$, $H^{n-3}(P;\mathbb{Z}_2)$ are isomorphic to $\mathbb{Z}_2$ with generators cd^{2s-1}, d^{2s-1}. Therefore

$$\bar{w}_{n-2} = \binom{2s+2s-1}{2s-1} \quad cd^{2s-1}$$

$$\bar{w}_{n-3} = \binom{2s+2s-1}{2s-1} \quad d^{2s-1}$$

The greatest power of 2 ν that divides the binomial coefficient $\binom{2s + 2s - 1}{2s-1}$ is given by

$$\begin{aligned}\nu &= \alpha(2s-1) + \alpha(2s) - \alpha(4s-1)\\ &= \alpha(s-1) + 1 + \alpha(s) - \alpha(s-1) - 2\\ &= \alpha(s) - 1\end{aligned}$$

It follows then that $\binom{2s + 2s - 1}{2s-1}$ is odd if and only if s is a power of 2. In this case the Stiefel-Whitney number $w_2 \cdot \bar{w}_{n-2}(P)[P]$ is non-zero. Otherwise $P \subset \mathbb{R}^{2n-2}$ by [Th, theorem 1.1]

We now compute the mod 2 cohomology of V using results and techniques of [Wa].

Lemma 59. <u>Let V be as in (57).</u> $H^*(V;\mathbb{Z}_2)$ <u>is the ring with generators z,c,a,d,b on dimensions 1,1,1,2,2 respectively, and relations</u>

$$z^2=0,\ c^{2^{\ell+1}} = c^{2^{\ell+1}-1}z,\ a^2 = az,\ d^{2^{m+\ell}+1}=0,\ b^{2^t+1} = 0.$$

Proof. The mod 2 cohomology of $P_1 \times P_2$ is the ring with generators c,d,a,b and relations $c^{2^{\ell+1}} = d^{2^{m+\ell}+1} = a^2 = b^{2^t+1} = o$. Since $A \times A$ acts trivially on $H^*(P_1 \times P_2; \mathbb{Z}_2)$, $\beta : V \to S^1$ induces an spectral sequence which collapses for dimensional reasons.

If $z = \beta^* u$ and c,d,a,b induce the classes of the same names in $H^*(P_1 \times P_2; \mathbb{Z}_2)$, then $H^*(V;\mathbb{Z}_2)$ has the additive basis $\{c^r d^s a^n b^m z^\varepsilon\}$.

Diagram (56) and [Wa; lemma 4] imply our assertion. $\square$

Notice that the induced homomorphism

$$i^* : H^*(Q_1 \times Q_2; \mathbb{Z}_2) \to H^*(V;\mathbb{Z}_2)$$

is given by

$i^*(x) = i^*(y) = z$, $i^*(a) = a$, $i^*(c) = c$, $i^*(d) = d$, $i^*(b) = b$.

Proof of proposition 57.

Since V is trivially embedded in $Q_1 \times Q_2$ then $i^* w(Q_1 \times Q_2) = w(V)$. We compute now some of these classes.

See [Wa, lemma 5].

$$w(Q_1) = (1+c+x)(1+c)^{2^{\ell+1}-2}(1+c+d)^{2^{m+\ell}+1}$$

$$= (1+c+x)(1+c^2+c^4+\ldots)(1+c+d)(1+d^{2^{m+\ell}})$$

$$= (1+c+x)(1+c^2)(1+c+d) + \text{elements of order} > 3.$$

Thus $w_1(Q_1) = x$, $w_2(Q_1) = cx+d$, $w_3(Q_1) = c^2x+dc+dx$.

$$\bar{w}(Q_1) = (1+c+c^2+\ldots+x+xc^2+xc^4+\ldots)(1+c^2+c^{2^{\ell+1}})(1+c+d)^{2^{m+\ell}-1}$$

$$= (1+c+x+c^{2^{\ell+1}})\Big(\sum_{i=0}^{2^{m+\ell}-1}(c+d)^i\Big)$$

$$= (1+c+x+c^{2^{\ell+1}})\Big(\sum_{i,j}\tbinom{j}{i}c^i d^{j-i}\Big) \qquad i,j \le 2^{m+\ell}-1$$

$H^{2a_1-2}(Q_1;\mathbb{Z}_2)$ is isomorphic to three copies of $\mathbb{Z}_2$ on generators $d^{2^{m+\ell}}c^{2^{\ell+1}-2}$, $d^{2^{m+\ell}}c^{2^{\ell+1}-3}x$, $d^{2^{m+\ell}-1}c^{2^{\ell+1}}$. It follows then that

$$\bar{w}_{2a_1-2}(Q_1) = d^{2^{m+\ell}-1}c^{2^{\ell+1}}.$$

On the other hand,

$$w(Q_2) = (1+a+y)(1+a+b)^{2^t+1}$$

$$= (1+a+y)(1+b^{2^t})(1+a+b)$$

$$= 1+y+b+(b_y+b_a) + b^{2^t} + b^{2^t}y$$

$$\text{and} \quad \bar{w}(Q_2) = (1 + a + y + a^2)\ (1 + a + b)^{2^t-1}$$
$$= (1 + a + y + a^2)\ \left(\sum_{i=0}^{2^t-1} (b^i + ib^{i-1}a + i(i-1)/2\ b^{i-2}a^2\right)$$

As $H^{2a_2-2}(Q_2;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by $b^{2^t}, b^{2^t-1}\ a^2$

we have

$$\bar{w}_{2a_2-2}(Q_2) = b^{2^t-1}\ a^2$$

It follows then that

$$\begin{aligned} w_3(V) &= i^*w_3(Q_1 \times Q_2) \\ &= i^*(c^2x+dc+(cx+d)y+xb+by+ab) \\ &= c^2z+dc+ab \end{aligned}$$

and as $\alpha(a_1)=\alpha a_2=2$ [M-P] imply $\bar{w}_{2a_i}(Q_i)= \bar{w}_{2a_i-1}(Q_i) = 0.$

Thus

$$\bar{w}_{2a_1+2a_2-3}(V)=i^*\bar{w}_{2a_1+2a_2-3}(Q_1\times Q_2) = 0$$

and

$$\begin{aligned} \bar{w}_{2a_1+2a_2-4}(V) &= i^*\bar{w}_{2a_1+2a_2-4}(Q_1\times Q_2) \\ &= i^*(\bar{w}_{2a_1-2}(Q_1)\times\bar{w}_{2a_2-2}(Q_2)) \\ &= d^{2^{m+\ell}-1}c^{2^{\ell+1}}b^{2^t-1}a^2 = 0 \end{aligned}$$

for $c^{2^{\ell+1}} = c^{2^{\ell+1}-1}z$, $a^2 = az$ and $z^2 = 0$. Proposition 57

now follows from [Th, Theorem 1.1] $\square$

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